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44

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BudkerINP 94-68

НОВОСИБИРСК

The Motion of Trapped Secondary Particles in a Storage Ring

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ABSTRACT

The longitudinal motion of secondary electrons and ions trapped by an electric circulating beam field in nonuniform magnetic field of the storage ring is studied analytically. The conditions for their reflection in the fringe field of the storage ring magnet and in the sign-alternating field of the undulator have been found. The calculations have been made for the probability of this reflection in the case of ion generated in a straight section, in the region of a zero magnetic field.

The secondary electrons or ions produced by ionization of the residual gas by a circulating beam can be trapped by its electric field and exist in the beam for a long time. The secondary particles moving in the nonuniform field of the magnetic storage ring system can, under certain conditions, change the direction of their longitudinal motion to the opposite one in the region of more powerful magnetic field and thus fall into a specific trap. This can result in the local increase in their density. As known [1–5], even substantially small concentrations of the secondary particles in the circulating beam have an influence on its dynamics and can be the reason of undesirable effects which give the change in its transverse sizes, the increase in transverse oscillations, and even its loss. For this reason, of importance is the knowledge of peculiarities of the motion of the secondary particles trapped by the electric circulating beam field.

Moreover, at present the methods for observation of parameters of the circulating fast-particle beam by registering the secondary particle beam are widely used [6, 7]. In this case, to correctly interpret the obtained results, we need rather exact data on conditions for storage of the secondary particles. This problem can be also important for realization in practice of proposals concerned with the methods for measuring the superhigh vacuum in cryogenic accelerators by registering the secondary ions from the beam or detecting bremsstrahlung photons produced in collisions of the circulating fast particles with the trapped secondary ions [8,9].

The problem of longitudinal motion of trapped particles in the nonuniform fringe field of the storage ring magnet was solved for the first time in [10], where the reflection criterion for heavy secondary particle produced in the region with the zero magnetic field was derived. It is interesting to study this problem in the more general case, and to try to derive the equations, describing the motion of the secondary particles by the more consistent way.

In plasma physics, the problems of this type are solved by the methods of drift approximation [11, 12]. This effective method is well developed and allows the obtaining of detailed information about the averaged motion of the particle in complicated configurations of the electric and magnetic fields.

Unfortunately, the possibilities of direct application of this method to solve the problem under consideration are restricted. The thing is that this method is based on the assumption of smallness of the Larmor radius value as compared with the characteristic sizes of the particle motion region, the smallness of the electrical field change in the interval having the size of the Larmor radius order, and the smallness of the Larmor rotation period as compared with the characteristic time of particle motion. In our case, these conditions are not always satisfied and can not be fulfilled in the region, where the magnetic field is close to zero as in straight sections of the storage ring. Nevertheless, we shall try to apply the drift approximation method to solve the problem posed, modifying it such that it could be applicable for the whole range of the problem parameter change, including the zero field region, which is of interest to us.

For this purpose, we shall consider the motion of the charged secondary particles in the electric field of the circulating beam moving in the X-direction. For the sake of simplicity, the circulating beam is supposed to be continuous. In the case of a sufficiently high frequency of the bunch passing, transition to a bunched beam can, in the first approximation, be carried out by replacing the real beam field by its averaged time value. Let the magnetic field $\mathbf{B}$ be directed along the Z-axis and $\vec{\nabla}|\mathbf{B}|$ be directed in the X-axis (Fig. 1). Such a geometry of the fields models the motion of the secondary particles near the storage ring magnet edge and their motion in the sign-alternating undulator field as well. To determine the main properties of this motion, we confine ourselves to solution of the linear problem, without consideration at this stage a series of important and interesting effects which are due to strong real nonlinearity of the circulating beam field and to its periodic dependence on time.

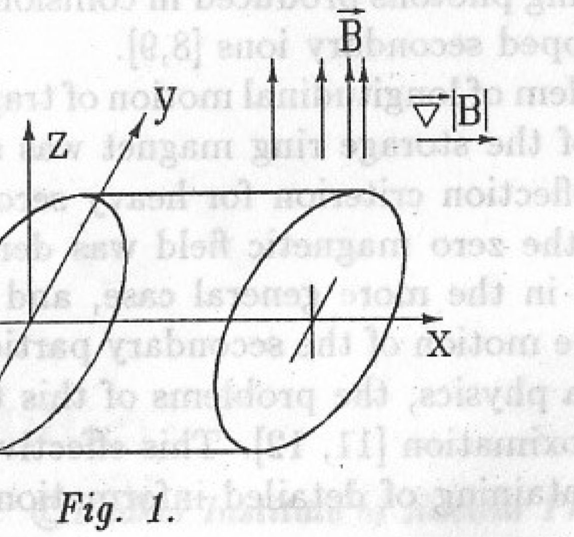

Fig. 1.

As known, the particle motion in such a geometry can be represented as a sum of quick oscillations of the particle relative to a certain point (leading center) which, in turn, slowly drifts in the given electric and magnetic fields. It is this drift that describes the averaged motion of the particle and is of the main interest to us.

Before deriving the modified drift equations describing this motion in the general case of nonuniform magnetic fields, we give a brief analysis of the particle motion in the simplest case, i.e., the case of a uniform magnetic field. The results of this analysis will be useful in further consideration.

The equation of particle motion in electric and magnetic fields has the form

$$M\frac{d^2\mathbf{R}}{dt^2} = e\mathbf{E} + \frac{e}{c}[\dot{\mathbf{R}} \times \mathbf{B}]\,, \tag{1}$$

where $\mathbf{R}$ is the radius-vector of the particle; e and M are its charge and mass, respectively; and c is the light velocity. The dependence of the electric field strength on the distance to the beam axis is assumed to be linear.

$$E_y = \frac{M}{e}\,\omega_y^2 Y\,,$$

$$E_z = \frac{M}{e}\,\omega_z^2 Z\,, \tag{2}$$

$$E_x = 0\,,$$

where ω_y and ω_z depend only on the form of the beam and on its density. Let us consider the particle motion in the plane (X, Y). With account of (2), the vector equation (1) in coordinates has the form

$$\ddot{X} = \omega_B \dot{Y}\,,$$

$$\ddot{Y} + \omega_y^2 Y = -\omega_B \dot{X}\,, \tag{3}$$

where $\omega_B = eB_z/Mc$.

The general solution of this system of equations which satisfies the arbitrary initial conditions $Y = Y_0$, $X = X_0$, $V_y = V_{y0}$, $V_x = V_{x0}$ when $t = 0$ is easily obtained. This solution has the following form

$$X = \bar{X} + \rho_x \sin(\Omega t + \Psi)\,, \qquad \dot{X} = \bar{V}_x + \rho_x \Omega \cos(\Omega t + \Psi)\,,$$

$$Y = \bar{Y} + \rho_y \cos(\Omega t + \Psi)\,, \qquad \dot{Y} = -\rho_y \Omega \sin(\Omega t + \Psi)\,, \tag{4}$$

where

$$\bar{X} = X_0 + \bar{V}_x t + \frac{\omega_B}{\Omega^2} V_{y0}\,, \tag{5a}$$

$$\bar{Y} = \frac{\omega_B^2}{\Omega^2}\left(Y_0 - \frac{V_{x0}}{\omega_B}\right), \tag{5b}$$

$$\rho_y = \left[\frac{\omega_B^4}{\Omega^4}\left(\frac{\omega_y^2}{\Omega_B^2} Y_0 + \frac{V_{x0}}{\omega_B}\right)^2 + \frac{V_{y0}^2}{\Omega^2}\right]^{1/2}, \tag{5c}$$

$$\rho_x = \frac{\omega_B}{\Omega} \rho_y\,, \tag{5d}$$

$$\bar{V}_x = \frac{\omega_y^2}{\Omega^2}(V_{x0} - \omega_B Y_0)\,, \tag{5e}$$

$$\Omega = \sqrt{\omega_B^2 + \omega_y^2}\,, \tag{5f}$$

$$\sin\Psi = -\frac{V_{y0}}{\rho_y \Omega}\,. \tag{5g}$$

Here $\bar{V}_x$ is meant as the particle drift velocity and, when $|\mathbf{B}| \to \infty$ it, as should be expected, transforms to the known expression $\bar{V}_x = c(E/B)$ for the particle drift velocity in the crossing electric and magnetic fields.

Let us now derive the drift equations which describe the secondary particle drift in the nonuniform magnetic field as applied to our particular problem. For this purpose, following the usual procedure, let us introduce the orthogonal system of unit vectors:

$$\mathbf{n}_0(\mathbf{R}) = \mathbf{B}/|\mathbf{B}|\,,$$

$$\mathbf{n}_1(\mathbf{R}) = [\mathbf{n}_2 \times \mathbf{n}_0]\,, \tag{6}$$

$$\mathbf{n}_2(\mathbf{R}) = [\mathbf{n}_0 \times \mathbf{n}_1]\,,$$

and hence,

$$[\mathbf{n}_1 \times \mathbf{n}_2] = \mathbf{n}_0.$$

Then, the particle velocity can be written as

$$\mathbf{V} = v_\parallel \mathbf{n}_0 + (V_1 + v_1\cos\theta)\mathbf{n}_1 + (V_2 + v_2\sin\theta)\mathbf{n}_2\,. \tag{7}$$

Generally speaking, the particle moves such that the values of V_1, v_1, V_2, v_2 and θ are the sums of the slow variables and quickly oscillating small terms. (From now on, by the slow variables, such variables are meant, the

relative change of which during the period characteristic of that of oscillations is small). However, since the latter ones are small, they can be redefined so that the quickly oscillating terms be contained only in V_1 and V_2 but only slow components be kept in the variables v_1, v_2 and $\dot\theta$. Taking into account the above fact, we shall consider that v_1, v_2 and $\dot\theta$ change slowly with time. Substituting expression (7) for velocity into (1) and multiplying scalarly the obtained vector equation by the vectors $\mathbf{n}_0$, $\mathbf{n}_1$ and $\mathbf{n}_2$, we obtain the following system of equations:

$$\frac{dV_1}{dt} + \dot{v}_1\cos\theta - \omega_B v_2 \sin\theta - v_1\dot\theta\sin\theta - \omega_B V_2 + \frac{v_\parallel^2}{B}\frac{\partial B_z}{\partial X} = \frac{e}{M}\mathbf{E}\mathbf{n}_1\,,$$

$$\frac{dV_2}{dt} + \dot{v}_2\sin\theta + \omega_B v_1\cos\theta + v_2\dot\theta\cos\theta + \omega_B V_1 = \frac{e}{M}\mathbf{E}\mathbf{n}_2\,, \tag{8}$$

$$\frac{dv_\parallel}{dt} - (V_1 + v_1\cos\theta)\frac{v_\parallel}{B}\frac{\partial B_z}{\partial X} = \frac{e}{M}\mathbf{E}\mathbf{n}_0\,.$$

Here, the point above the variable means, as usually, the time derivative $\dot{\mathbf{F}} = \frac{\partial \mathbf{F}}{\partial t} + (\mathbf{V}\vec\nabla)\mathbf{F}$ ($\mathbf{F}$ is the arbitrary vector function of coordinates and time). In deriving this system, we use an assumption of smallness of the value of $\frac{z}{B}\frac{\partial B_z}{\partial X}$, as compared with unity, which is one of criteria for validity of the results to be obtained.

After sufficiently bulky but not very complicated calculations which are briefly presented in Appendix, from the above equations we can obtain the desired system of drift equations describing the motion of the leading center:

$$\frac{d^2\bar X}{dt^2} - \omega_B\frac{d\bar Y}{dt} - \frac{1}{2\theta}v_1 v_2\frac{d\omega_B}{dX} = 0\,, \tag{9a}$$

$$\frac{d^2\bar Y}{dt^2} + \omega_B\frac{d\bar X}{dt} + \omega_y^2\bar Y = 0\,, \tag{9b}$$

$$\frac{v_2^2}{\Omega} = \text{const}\,, \tag{9c}$$

$$v_1 = \frac{\omega_B}{\Omega}v_2\,, \tag{9d}$$

$$\dot\theta = \Omega = \sqrt{\omega_y^2 + \omega_B^2}\,, \tag{9e}$$

$$\rho_1 = \frac{1}{\Omega}v_1\,, \tag{9f}$$

$$\rho_2 = \frac{1}{\Omega}v_2\,. \tag{9g}$$

As follows from the derivation of system (9), it is valid if the following conditions are simultaneously satisfied:

$$\frac{z}{B}\left|\frac{\partial B_z}{\partial X}\right| \ll 1 \qquad (10a)$$

$$\frac{\rho_1}{B}\left|\frac{\partial B_z}{\partial X}\right| \ll 1 \qquad (10b)$$

the latter of which can be rewritten, using (9e) and (9f) as

$$\left|\frac{v_2}{\Omega^2}\frac{d\omega_B}{dX}\right| \ll 1\,, \qquad (10c)$$

The other condition for validity of (9) is the one of smallness of the oscillation frequency change during the time interval of an order of their period

$$\left|\frac{1}{\Omega^2}\frac{d\Omega}{dt}\right| \ll 1,$$

which can be rewritten as follows:

$$\left|\frac{\omega_B}{\Omega^3}V_x\frac{d\omega_B}{dX}\right| \ll 1\,. \qquad (10d)$$

It is obvious that to satisfy all the above conditions, at least the logarithmic derivative $\frac{1}{B}\frac{\partial B_z}{\partial X}$ should be restricted over the whole region of the particle motion. The latter condition is always fulfilled in the region of the fringe magnet field and a straight section, however, it is not satisfied in the sign-alternating undulator field within a small region of the change in the longitudinal coordinate X, in the vicinity of the point, where the field $\mathbf{B}$ is equal to zero. It is natural that this interval should be excluded from our analysis.

It should be noted that unlike the usual method of drift approximation which is valid only in the region of rather high magnetic fields, conditions (10a)–(10d) can be satisfied, hence system (9) is valid for extremely low fields, including the region of the zero field.

To finish the analysis of system (9), we should consider a somewhat unexpected fact. In the general form, the drift equations [9,10] comprise the term related to the "gradient" drift which is proportional to the squared velocity component directed perpendicular to the magnetic field and the term related to the "centrifugal" drift which is proportional to the squared velocity component parallel to the magnetic field. Our equations do not comprise the latter

term, and the drift equations are independent of the particle oscillations in the Z-direction. This fact is explained as follows. If we consider the particle motion in the earlier used system of coordinates $\mathbf{n}_0$, $\mathbf{n}_1$ and $\mathbf{n}_2$, then it is revealed that $\langle F_{cf}\rangle - \langle eE_1\rangle = 0$ in the case of the linear dependence of the electric field strength on z. Here F_{cf} is the centrifugal force value associated with the longitudinal motion of the particle relative to the magnetic field and E_1 is the electric field component parallel to $\mathbf{n}_1$ (the averaging is made by the period of vertical oscillations). Thus, this fact is associated just with the property of the chosen configuration of the electric and magnetic fields.

Before solving system (9), we first should note that the second derivative in Eq.(9b) can be neglected because the drift velocity component in the Y-direction is mainly determined by the magnetic field gradient and its acceleration is determined by the squared gradient. To neglect the second derivative in Eq.(9a), especially in the case, where the magnetic field is small, there are no such grounds as above, so it should be kept. Taking into account this fact and using relations (9c)–(9g), we can rewrite Eqs.(9a)–(9b) as follows:

$$\frac{d^2X}{dt^2} - \omega_B\frac{dY}{dt} + \frac{1}{2}\frac{\omega_B}{\Omega}\left(\frac{v_2(0)^2}{\Omega(0)}\right)\frac{d\omega_B}{dX} = 0\,, \qquad (11)$$

$$\omega_B\frac{dX}{dt} = -\omega_y^2 Y\,,$$

where $v_2(0)$ and $\Omega(0)$ are the values of the corresponding variables at the initial moment of time. (Since we use only the averaged variables, the symbol of averaging of the variables X and Y is omitted from now on.)

Excluding time from the system of equations (11), we obtain the following equation describing the form of the drift trajectory

$$\frac{d}{dX}\left[\left(\frac{\omega_y^2}{\omega_B}Y\right)^2\right] = -\left[\frac{2}{\Omega}\left(\frac{\omega_y^2}{\omega_B}Y\right)^2 + \frac{\omega_y^2}{\Omega^2}\frac{v_2(0)^2}{\Omega(0)}\right]\frac{\omega_B}{\Omega}\frac{d\omega_B}{dX} \qquad (12)$$

To obtain the result which is independent of the specific dependence of the magnetic field on the coordinate X, we jump from the independent coordinate X to the independent coordinate Ω

$$\frac{d}{d\Omega}\left[\left(\frac{\omega_y^2}{\omega_B}Y\right)^2\right] = -\frac{2}{\Omega}\left(\frac{\omega_y^2}{\omega_B}Y\right)^2 - \frac{\omega_y^2}{\Omega^2}\frac{v_2(0)^2}{\Omega(0)}\,. \qquad (13)$$

The solution of this equation, which satisfies the initial condition $Y = Y_0$ when $\Omega = \Omega(0)$, has the form

$$Y^2 = \frac{\Omega(0)^2}{\Omega^2}\left[\frac{\omega_B^2}{\omega_B(0)^2}Y_0^2 - \frac{\omega_B^2}{\omega_y^2}\frac{v_2(0)^2}{\Omega(0)^2}\left(\frac{\Omega}{\Omega(0)}-1\right)\right], \qquad (14)$$

where $\omega_B(0)$ is the initial value of the frequency ω_B. By expressing the values of Y_0 and $v_2(0)$ in terms of the initial velocities and coordinates of the secondary particle production using relations (5b), (5c), and (5g), we finally obtain the following equation for the drift trajectory of the particle:

$$Y^2 = \frac{\Omega(0)^2}{\Omega^2}\left\{\frac{\omega_B^2\omega_B(0)^2}{\Omega(0)^4}\left(y_0 - \frac{V_{0x}}{\omega_B(0)}\right)^2 - \frac{\omega_B^2}{\omega_y^2}\left[\frac{\omega_B(0)^4}{\Omega(0)^4}\left(\frac{\omega_y^2}{\omega_B(0)^2}y_0 + \frac{V_{0x}}{\omega_B(0)}\right)^2 + \frac{V_{0y}^2}{\Omega(0)^2}\right]\left[\frac{\Omega}{\Omega(0)}-1\right]\right\} \qquad (15a)$$

which, in the case of the secondary particle production in the straight section ($B(0) \to 0$) takes the form (provided that $y_0 \neq 0$)

$$Y^2 = \frac{\omega_B^2}{\omega_y^2}\left[\frac{V_{x0}^2}{\omega_y^2} - \left(y_0^2 + \frac{V_{y0}^2}{\omega_y^2}\right)\left(\frac{\Omega}{\Omega(0)}-1\right)\right] \qquad (15b)$$

(y_0 is a point where the secondary particle was created).

Figure 2 presents the drift trajectories of ion motion, which are calculated by the formula (15b). For the sake of definition, we take the numerical values used for computer simulation in [10] ($\omega_y = 3.9\cdot 10^6\ s^{-1}$, $V_{0x} = 2.0\cdot 10^5$ cm/s, $V_{0y} = 0$, $M = 28$). It is seen that the results are in good agreement.

For practice, it is important to know the conditions under which the secondary particle, moving in the nonuniform magnetic field, reflects from the fringe field of bending magnets of the storage ring, thus being trapped.

It follows from Eq.(15a) that the particle reflection criterion is

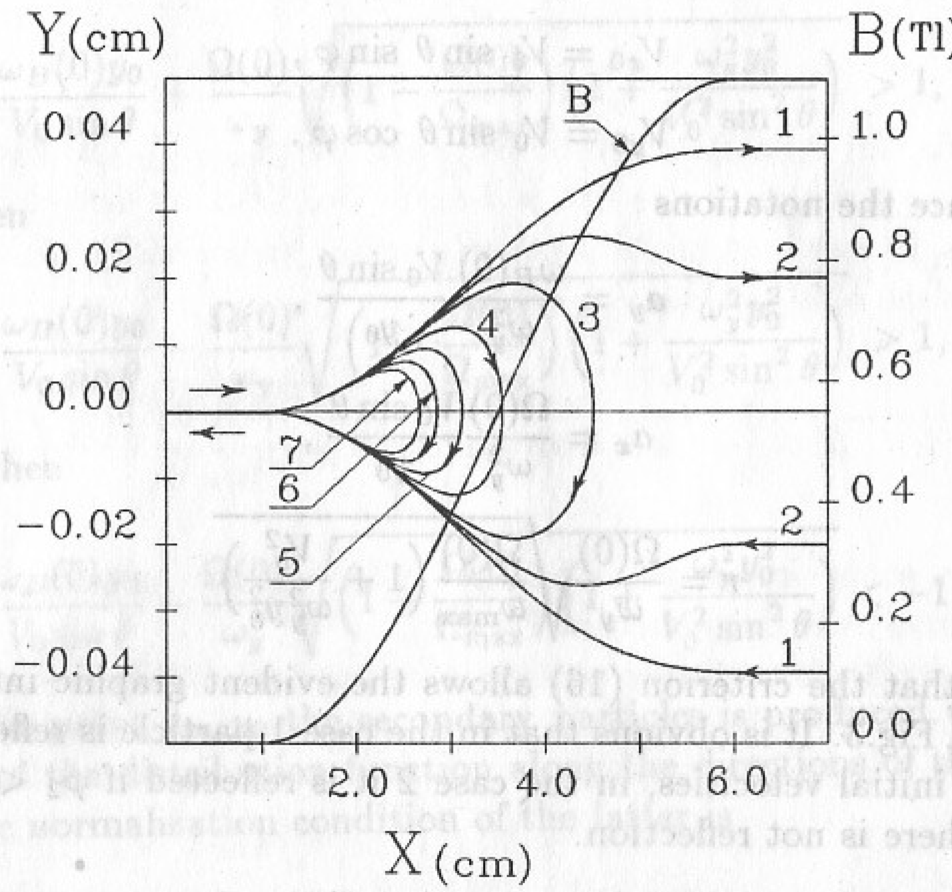

Fig. 2. Ion drift trajectories in magnetic field: 1) $y_o = 0.05$ cm; 2) $y_0 = 0.075$ cm; 3) $y_0 = 0.10$ cm; 4) $y_0 = 0.15$ cm, 5) $y_0 = 0.20$ cm; 6) $y_0 = 0.25$ cm; 7) $y_0 = 0.30$ cm.

$$\frac{1}{\omega_B(0)^2}\left(\frac{\omega_B(0)}{\Omega(0)}\right)^4\left(y_0 - \frac{V_{x0}}{\omega_B(0)}\right)^2 - \frac{1}{\omega_y^2}\left\{\frac{\omega_B(0)^4}{\Omega(0)^4}\left(\frac{\omega_y^2}{\omega_B(0)^2}y_0 + \frac{V_{x0}}{\omega_B(0)}\right)^2 + \frac{V_{y0}^2}{\Omega(0)^2}\right\}\left(\frac{\Omega_{max}}{\Omega(0)}-1\right) < 0,$$

where Ω_{max} is the value of the quantity Ω at maximum magnetic field. After simple transformations this expression can be presented in a more convenient form:

$$\left(1+\frac{\omega_B(0)}{\omega_y^2}\frac{V_{x0}}{y_0}\right)^2 + \frac{\Omega(0)^2}{\omega_y^4}\frac{V_{y0}^2}{y_0^2} > \frac{\Omega(0)^2}{\omega_y^2}\frac{\Omega(0)}{\Omega_{max}}\left(1+\frac{V_{0x}^2+V_{0y}^2}{\omega_y^2 y_0^2}\right) \qquad (16)$$

when $\omega_B(0) = 0$ and $\frac{\omega_{B\max}}{\omega_y} \ll 1$ this expression takes the form $\frac{V_{0x}}{y_0} < \frac{1}{\sqrt{2}}\omega_{B_{\max}}$ which was obtained in [10]. If at the moment of its production the secondary particle has the velocity V_0, then in the spherical coordinates whose polar axis coincides with the Z-axis the components of the initial particle velocity entering this equation can be written in the form

$$V_{x0} = V_0 \sin\theta \sin\varphi ,$$

$$V_{y0} = V_0 \sin\theta \cos\varphi . \tag{17}$$

If to introduce the notations

$$a_y = \frac{\omega_B(0)\, V_0 \sin\theta}{\omega_y^2 \qquad y_0},$$

$$a_x = \frac{\Omega(0)\, V_0 \sin\theta}{\omega_y^2 \qquad y_0}, \tag{18}$$

$$r = \frac{\Omega(0)}{\omega_y}\sqrt{\frac{\Omega(0)}{\omega_{\max}}\left(1+\frac{V_0^2}{\omega_y^2 y_0^2}\right)}$$

we can see that the criterion (16) allows the evident graphic interpretation presented in Fig.3. It is obvious that in the case 1 particle is reflected at any ratio of the initial velocities, in the case 2 it is reflected if $\varphi_2 < \varphi < \varphi_1$, in the case 3 there is not reflection.

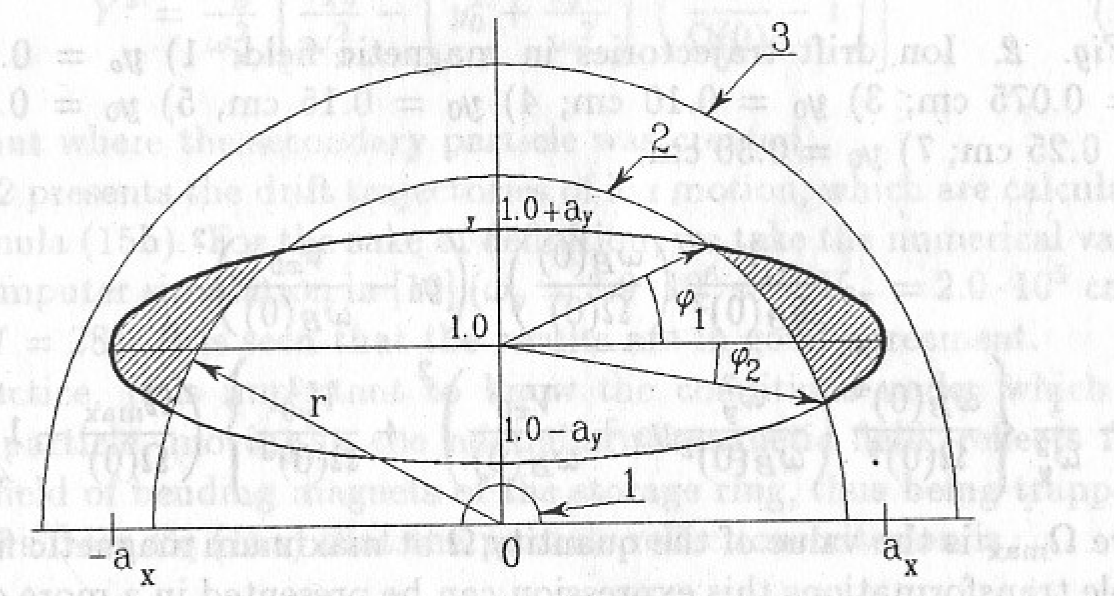

Fig. 3.

From Eq. (16), we can obtain the following expression for the values of φ_1 and φ_2:

$$\varphi_{1,2} = \arcsin\frac{\omega_B(0) y_0}{V_0 \sin\theta} \pm \frac{\Omega(0)}{\omega_y}\sqrt{\left(1-\frac{\Omega(0)}{\Omega_{\max}}\right)\left(1+\frac{\omega_y^2 y_0^2}{V_0^2 \sin^2\theta}\right)}, \tag{19a}$$

which, as follows from the analysis of limiting cases should be additionally determined as follows:
$\varphi_1 = \frac{\pi}{2}$ when

$$\frac{\omega_B(0) y_0}{V_0 \sin\theta} + \frac{\Omega(0)}{\omega_y}\sqrt{\left(1-\frac{\Omega(0)}{\Omega_{\max}}\right)\left(1+\frac{\omega_y^2 y_0^2}{V_0^2 \sin^2\theta}\right)} > 1, \tag{19b}$$

$\varphi_2 = \frac{\pi}{2}$ when

$$\frac{\omega_B(0) y_0}{V_0 \sin\theta} - \frac{\Omega(0)}{\omega_y}\sqrt{\left(1-\frac{\Omega(0)}{\Omega_{\max}}\right)\left(1+\frac{\omega_y^2 y_0^2}{V_0^2 \sin^2\theta}\right)} > 1, \tag{19c}$$

$\varphi_2 = -\frac{\pi}{2}$ when

$$\frac{\omega_B(0) y_0}{V_0 \sin\theta} - \frac{\Omega(0)}{\omega_y}\sqrt{\left(1-\frac{\Omega(0)}{\Omega_{\max}}\right)\left(1+\frac{\omega_y^2 y_0^2}{V_0^2 \sin^2\theta}\right)} < -1. \tag{19d}$$

Let in the point x_0, y_0 the secondary particles is produced with the velocity V_0 and the distribution function along the directions of the velocities $f(\theta,\varphi)$. The normalization condition of the latter is

$$\int_0^{\pi}\int_0^{2\pi} f(\theta,\varphi)\sin\theta,\, d\theta,\, d\varphi .$$

The probability of particle reflection in the region of the strong magnetic field can be expressed as follows:

$$P = \frac{1}{\pi}\int_0^{\pi/2} d\theta \sin\theta \int_{\varphi_2}^{\varphi_1} f(\theta,\varphi)\, d\varphi , \tag{20}$$

where φ_1 and φ_2 are found from (19). Using this expression, we can easily calculate the desired probability for any particular case.

In this study, we shall confine ourselves to the simple, however, important, case where the particle forms in the straight section, in the region of the zero magnetic field. In this case, Eq.(19a) is written in the form

$$\varphi_{1,2} = \pm\arcsin\sqrt{\left(1-\frac{\Omega(0)}{\Omega_{\max}}\right)\left(1+\frac{\omega_y^2 y_0^2}{V_0^2 \sin^2\theta}\right)} , \tag{21}$$

Let us introduce the dimensionless variables $\eta = \sqrt{1 - (\Omega(0)/\Omega_{\max})}$ and $\xi = V_0/\omega_y y_0$. The physical sense of the latter is obvious if it is written in the form

$$\xi^2 = \frac{W_0}{e\varphi_{\text{beam}}}\left(\frac{\sigma_0}{y_0}\right)^2,$$

where W_0 is the initial energy of the particle, φ_{beam} is the beam potential, and σ_y is the transverse size of the beam. The velocity distribution function of particles is assumed to be isotropic, what is probably valid at least for ions. Equation (20) is then written as

$$P = \frac{2}{\pi}\int_0^{\pi/2} \sin\theta \arcsin\left[\eta\sqrt{1 + \frac{1}{\xi^2 \sin^2\theta}}\right] d\theta. \qquad (22)$$

In this case, some properties of particle reflection can be determined without obvious numerical integration of Eq.(22).

1. Since in the case under consideration $a_y = 0$, it follows from Fig.3 that the probability of reflection is equal to unity in the parameter domain, in which $r < 1$ whence $P = 1$ when $\xi < \frac{\eta}{\sqrt{1-\eta^2}}$.

2. It is easily obtained that with the large values of ξ the probability of particle reflection is determined by the following simple expression:

$$P \simeq \frac{2}{\pi} \arcsin \eta.$$

Integrating numerically expression (22), we obtain the result which is illustrated graphically in the coordinates η, ξ in Fig.4a. The parameter domain in which the probability of reflection is equal to unity is dashed.

It is interesting to make the calculation for the probability of particle reflection which is averaged in the particle production coordinate y_0 because it is this value which is observed in the experiment.

We shall introduce a new dimensionless variable $\xi_1 = V_0/\omega_y \sigma_y$. Let the distribution function of the secondary particle along the coordinate of their production be $g(y_0/\sigma_y)$ with the appropriate normalization. The expression for the probability of reflection which is averaged in the coordinate y can be written as

$$\langle P \rangle = \frac{2}{\pi}\int_0^1\int_0^{\pi/2} g\left(\frac{y_0}{\sigma_y}\right) \arcsin\left[\eta\sqrt{1 + \frac{1}{\xi_1^2 \sin^2\theta}\left(\frac{y_0}{\sigma_y}\right)^2}\right] d\left(\frac{y_0}{\sigma_y}\right) d\theta \qquad (23)$$

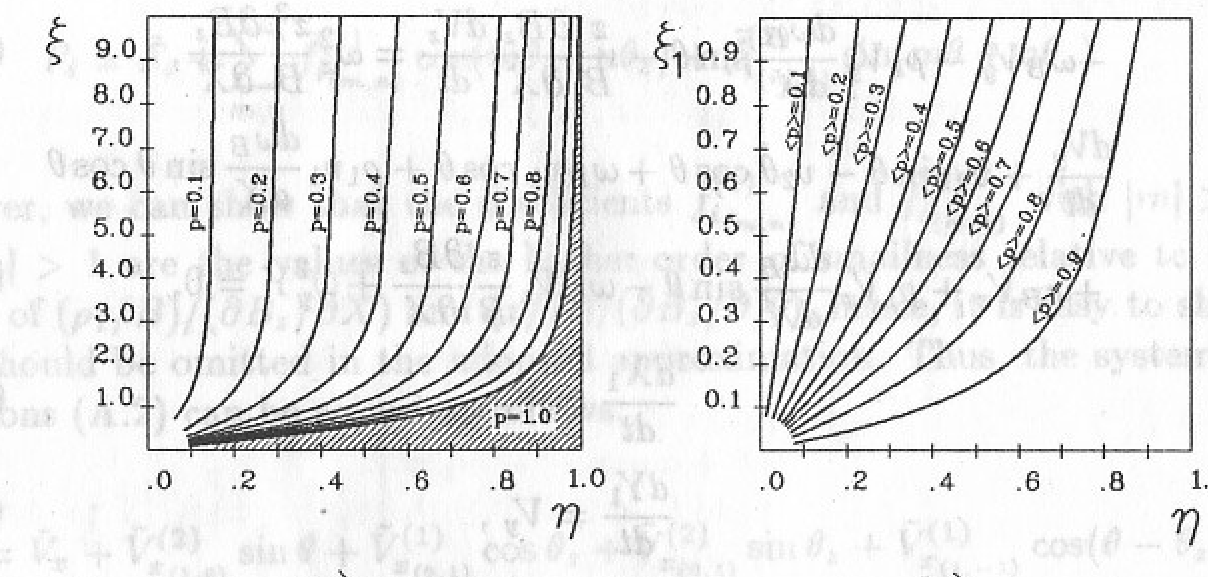

Fig. 4.

on the assumption that $g(y_0/\sigma_y)$ is constant. Integrating gives the result presented in Fig.4b

Thus, as follows from the example given, the probability of the secondary particle reflection in the region of the strong magnetic field is rather high. This can one of the reasons of the increase in local density of the particles.

Appendix. Derivation of the system of drift equations

Transition to the laboratory system of coordinates performed by the formulas

$$v_{\|} = V_z + \frac{z}{B}\frac{\partial B_z}{\partial X} V_x$$

$$V_1 = V_x - \frac{z}{B}\frac{\partial B_z}{\partial X} V_z \qquad (A.1)$$

$$V_2 = V_2\,,$$

where V_x and V_y are the corresponding components of the leading center velocity (determined analogously to V_1 and V_2 in (7)), reduces system (8) to the following form:

$$\frac{dV_z}{dt} + \frac{1}{B}\frac{\partial B_z}{\partial X}\frac{dV_x}{dt} z + \omega_z^2 z = 0, \qquad (A.2a)$$

$$\frac{dV_x}{dt} + \dot{v}_1 \cos\theta - v_1 \dot{\theta} \sin\theta - \omega_B v_2 \sin\theta - \rho_1 v_2 \frac{d\omega_B}{dX} \sin^2\theta$$

$$-\omega_B V_y - \rho_1 V_y \frac{d\omega_B}{dX} \sin\theta - \frac{z}{B}\frac{\partial B_z}{\partial X}\frac{dV_z}{dt} = \omega_z^2 \frac{z^2}{B}\frac{\partial B_z}{\partial X}, \quad (A.2b)$$

$$\frac{dV_y}{dt} + \dot{v}_2 \sin\theta - v_2 \dot{\theta} \cos\theta + \omega_B v_1 \cos\theta + \rho_1 v_1 \frac{d\omega_B}{dX} \sin\theta \cos\theta$$

$$+\omega_B V_x + \rho_1 V_x \frac{d\omega_B}{dX} \sin\theta - \omega_B V_z \frac{z}{B}\frac{\partial B_z}{\partial X} + \omega_y^2 Y = 0, \quad (A.2c)$$

$$\frac{dX_1}{dt} = V_x, \quad (A.2d)$$

$$\frac{dY_1}{dt} = V_y, \quad (A.2e)$$

$$v_1 = \rho_1 \dot{\theta}, \quad (A.2f)$$

$$v_2 = \rho_2 \dot{\theta}, \quad (A.2g)$$

where ω_B and $(1/B)/(\partial B_z/\partial X)$ are taken at the point of the leading center. To obtain this system, we use relations (2) and expansion of the magnetic field into a series $B_z = (X + \Delta X) = B_z(X) + \frac{\partial B_z}{\partial X}\Delta X$. The values of $(z/B)/(\partial B_z/\partial X)$ and $(\rho_1/B)/(\partial B_z/\partial X)$ are assumed to be small and only the terms of the first order of smallness are kept. In this case, we ought to introduce the additional variables X_1, Y_1, ρ_1 and ρ_2 which are defined, analogously to (7), by the following relations:

$$X = X_1 + \rho_1 \sin\theta,$$

$$Y = Y_1 + \rho_2 \sin\theta, \quad (A.3)$$

where X and Y are the coordinates of particle position, ρ_1 and ρ_2 are the slow variables. The additional equations (A.2d)–(A.2g) entering the initial system are the obvious consequence of (A.3).

As is mentioned above, the values of V_x, V_y, X_1 and Y_1 are a sum of the slow and fast terms. To distinguish this fact, we write each of these variables in the form

$$F_i = \bar{F}_i + \tilde{F}_i,$$

where F_i is any of the above-mentioned functions is meant, $\bar{F}$ and $\tilde{F}$ are the slow and fast parts of the function, respectively.

In principle, the system of Eqs.(A.2) should be solved in the following form:

$$F_i = \bar{F}_i + \sum_{m,n} \tilde{f}^{(1)}_{i_{(m,n)}} \cos(m\theta + n\theta_z) + \tilde{f}^{(2)}_{i_{(m,n)}} \sin(m\theta + n\theta_z).$$

However, we can show that the coefficients $\tilde{f}^{(1)}_{i_{(m,n)}}$ and $\tilde{f}^{(2)}_{i_{(m,n)}}$ with $|m| > 1$ and $|n| > 1$ are the values of the higher order of smallness relative to the values of $(\rho_1/B)/(\partial B_z/\partial X)$ and $(z/B)/(\partial B_z/\partial X)$, hence, it is easy to show they should be omitted in the adopted approximation. Thus, the system of equations (A.2) can be solved as follows:

$$V_x = \bar{V}_x + \tilde{V}^{(2)}_{x(1,0)} \sin\theta + \tilde{V}^{(1)}_{x(0,1)} \cos\theta_z + \tilde{V}^{(2)}_{x(0,1)} \sin\theta_z + \tilde{V}^{(1)}_{x(1,-1)} \cos(\theta - \theta_z)$$

$$+\tilde{V}^{(2)}_{x(1,-1)} \sin(\theta - \theta_z) + \tilde{V}^{(1)}_{x(1,1)} \cos(\theta + \theta_z) + \tilde{V}^{(2)}_{x(1,1)} \sin(\theta + \theta_z),$$

$$V_y = \bar{V}_y + \tilde{V}^{(1)}_{y(1,0)} \cos\theta + \tilde{V}^{(1)}_{y(0,1)} \cos\theta_z + \tilde{V}^{(2)}_{y(0,1)} \sin\theta_z + \tilde{V}_{y(1,-1)} \cos(\theta - \theta_z)$$

$$+\tilde{V}^{(2)}_{y(1,-1)} \sin(\theta - \theta_z) + \tilde{V}^{(1)}_{y(1,1)} \cos(\theta + \theta_z) + \tilde{V}^{(2)}_{y(1,1)} \sin(\theta + \theta_z),$$

$$X = \bar{X} + \tilde{X}^{(1)}_{(1,0)} \cos\theta + \tilde{X}^{(1)}_{(0,1)} \cos\theta_z + \tilde{X}^{(2)}_{(0,1)} \sin\theta_z + \tilde{X}^{(1)}_{(1,-1)} \cos(\theta - \theta_z)$$

$$+\tilde{X}^{(2)}_{(1,-1)} \sin(\theta - \theta_z) + \tilde{X}^{(1)}_{(1,1)} \cos(\theta + \theta_z) + \tilde{X}^{(2)}_{(1,1)} \sin(\theta + \theta_z),$$

$$Y_x = \bar{Y}_x + \tilde{Y}^{(2)}_{(1,0)} \sin\theta + \tilde{Y}^{(1)}_{(0,1)} \cos\theta_z + \tilde{Y}^{(2)}_{(0,1)} \sin\theta_z + \tilde{Y}^{(1)}_{(1,-1)} \cos(\theta - \theta_z)$$

$$+\tilde{Y}^{(2)}_{(1,-1)} \sin(\theta - \theta_z) + \tilde{Y}^{(1)}_{(1,1)} \cos(\theta + \theta_z) + \tilde{Y}^{(2)}_{(1,1)} \sin(\theta + \theta_z),$$

$$Z = Z^{(1)}_{(0,1)} \cos\theta_z + Z^{(2)}_{(0,1)} \sin\theta_z + Z^{(1)}_{(1,0)} \cos\theta + Z^{(2)}_{(1,0)} \sin\theta$$

$$+Z^{(1)}_{(1,-1)} \cos(\theta - \theta_z) + Z^{(2)}_{(1,1)} \sin(\theta - \theta_z) + Z^{(1)}_{(1,1)} \cos(\theta + \theta_z) + Z^{(2)}_{(1,1)} \sin(\theta + \theta_z),$$

where the values with the sign $\sim$ are also the values of the first order of smallness. Substituting these expressions into (A.2), multiplying the expression obtained from (A.2a)–(A.2c) by 1, $\sin\theta$ and $\cos\theta$, averaging them in the time interval which is much larger than $\tau_1 = 1/\dot\theta$ and $\tau_2 = 1/\dot\theta_z$ (the resonance cases $\dot\theta \pm \dot\theta_z = 0$ and $2\theta \pm \theta_z$ is ignored), and neglecting the values of the higher order of smallness, we obtain

$$\frac{d^2\tilde X}{dt^2} - \omega_B \frac{d\tilde Y}{dt} - \frac{1}{2\dot\theta} v_1 v_2 \frac{d\omega_B}{dX} = 0, \qquad (A.4a)$$

$$\frac{d^2\tilde Y}{dt^2} + \omega_B \frac{d\tilde X}{dt} + \omega_y^2 \tilde Y = 0, \qquad (A.4b)$$

$$v_1\dot\theta + \omega_B v_2 = 0, \qquad (A.4c)$$

$$v_2\dot\theta + \omega_B v_1 - \frac{\omega_y^2}{\dot\theta} v_2 = 0, \qquad (A.4d)$$

$$\tilde V_x \dot\theta + \dot v_1 - \omega_B \tilde V_y = 0, \qquad (A.4e)$$

$$-\tilde V_y \dot\theta + \dot v_2 + \omega_B \tilde V_x + \omega_y^2 \tilde Y + \tilde V_x \frac{v_1}{\dot\theta} \frac{d\omega_B}{dX} = 0, \qquad (A.4f)$$

$$\dot{\tilde Y} - \rho_2 \dot\theta = v_2, \qquad (A.4g)$$

$$\dot{\tilde X} + \rho_1 \dot\theta = v_1, \qquad (A.4h)$$

$$\tilde Y \dot\theta + \dot\rho_2 = \tilde V_y, \qquad (A.4i)$$

$$-\tilde X \dot\theta + \dot\rho_1 = \tilde V_x, \qquad (A.4k)$$

For brevity, we redenote the variables

$$\tilde V^{(2)}_{x(1,0)} \Rightarrow \tilde V_x;\ \tilde V^{(1)}_{y(1,0)} \Rightarrow \tilde V_y;\ \tilde X^{(1)}_{(1,0)} \Rightarrow \tilde X;\ \tilde Y^{(2)}_{(1,1)} \Rightarrow \tilde Y.$$

The first two equations of system (A.4) are the desired drift equations describing the motion of the leading center of the secondary particle. However, to use these equations in practice, it is necessary to find the unknown functions entering (A.4a).

From the compatibility condition of a pair of equations (A.4c) and (A.4d)

$$\begin{vmatrix} \dot\theta & \omega_B \\ \omega_B & \dot\theta - \frac{\omega_y^2}{\dot\theta} \end{vmatrix} = 0$$

we obtain the necessary relation

$$\dot\theta^2 = \Omega^2 = \omega_y^2 + \omega_B^2, \qquad (A.5)$$

which coincides with (5f), and from (A.4c) or (A.4d)

$$v_1 = -\frac{\omega_B}{\Omega} v_2, \qquad (A.6)$$

which also coincides with the analogous expression of system (5) from any of the above equations.

Let us now consider individually Eqs. (A.4e) and (A.4f).

$$\dot v_1 = \omega_B \tilde V_y - \Omega \tilde V_x, \qquad (A.7a)$$

$$\dot v_2 = \Omega \tilde V_y - \omega_B \tilde V_x - \omega_y^2 \tilde Y - \tilde V_x \frac{v_1}{\Omega} \frac{d\omega_B}{dX}. \qquad (A.7b)$$

Substituting $\tilde Y$ from (A.4i) into (A.7b), we obtain

$$v_2 = \frac{\omega_B}{\Omega} \dot v_1 - \frac{\omega_y^2}{\Omega^2} \dot v_2 + \frac{\Omega \omega_y^2}{\Omega^3} v_2 - \frac{v_1}{\Omega} \frac{d\omega_B}{dX}.$$

Substituting expression (A.6) into it, we obtain the last necessary relation

$$\frac{d}{dt}\left(\frac{v_2^2}{\Omega}\right) = 0, \quad \text{or} \quad \frac{v_2^2}{\Omega} = \text{const}.$$

This expression is the analog of the adiabatic invariant $v_\perp^2/B$=const from the standard drift theory and transforms to it when $B \to \infty$.

Acknowledgements.

I would like to express my deep gratitude to A.V. Burov, V.G. Davidovsky and D.V. Pestrikov for their interest to this study, fruitful discussions and helpful remarks.

References

[1] *B.V.Chirikov.* "Stability of a Partially Compensated Electron Beam". Atomnaja Energia, v.19, N3, p.239 (1965) (in Russian).

[2] *L.J.Laslett, A.M.Sessler, D.Mohl.* "Transverse Two-Stream Instability in the Presence of Strong Spacies-Species and Image Forces." Preprint LBL-1072 (1972).

[3] *P.R.Zenkevich, D.G.Koshkarev.* "Coupling Resonances of the Trasverse Oscillations of Two Circular Beams", Particle Accelerators, v.3, 1 (1972).

[4] *G.I.Budker, G.I.Dimov, G.V.Rosljakov, V.E.Chuprijanov, V.G. Shamovsky.* "Proton Accumulation in the Storage Ring at a Current Exceeding the Space Charge Limit", Preprint INP 77-51, Novosibirsk (1977) (in Russian).

[5] *Y.Baconnier.* "Neutralization of Accelerator Beams by Ionization of the Residual Gas", Proceedings of CERN accelerator school, v.2, p.525 Geneva, 1994.

[6] *Tadamichi Kawakubo et.al.* "Non-destructive Beam Profile Monitors in KEK Proton Synchrotron", Proceedings of the Workshop Advanced Beam Instrumentation, v.1, p.131, (1991) KEK, Tsukuba, Japan.

[7] *R.L.Witkover.* "New Beam Instrumentation in ASS Booster", Proceedings of the Workshop Advanced Beam KEK Proton Synchrotron", v.1 p.50, (1991) KEK, Tsukuba, Japan.

[8] *P.F.Tavares.* "Using the Electron Current from Ionization of the Residual Gas by the Proton Beam to Measure the Pressure in the CERN LHC cold bore", LNLS Note 93, September 1993.

[9] *P.F.Tavares.* "Bremsstrahlung Detection of Ions Trapped in the EPA Electron Beam", Particl Accelerators, v.43,(1-2), p.107.

[10] *Y.Miyahara, K.Takayama,G.Horikoshi.* "Dinamical Analysis on the Longitudinal Motion of Trapped Ions", Nucl. Instrum. and Methods, A270, N2, p.217, 1988.

[11] *Sivukhin.* "Drift Theory of Motion of a Charged Particle in Electromagnetic Fields", Voprosy teorii plasmi, gosatomizdat N1 p.7, 1963 (in Russian).

[12] *Morozov.* "The Motion of Charged Particles in Electromagnetic Fields", Voprosy teorii plasmi, gosatomizdat N2, p.177, 1963 (in Russian).

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Движение вторичных частиц в накопителе

BudkerINP 94-68

Ответственный за выпуск С.Г. Попов
Работа поступила 29 июня 1994 г.

Сдано в набор 29.06. 1994 г.
Подписано в печать 14 июля 1994 г.
Формат бумаги 60×90 1/16 Объем 1,4 печ.л., 1,1 уч.-изд.л.
Тираж 200 экз. Бесплатно. Заказ N 68

Обработано на IBM PC и отпечатано на
ротапринте ИЯФ им. Г.И. Будкера СО РАН,
Новосибирск, 630090, пр. академика Лаврентьева, 11.