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НОВОСИБИРСК

Plasma Equilibrium at Intense Injection of Neutral Beams

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Abstract

The injection of atomic beams into relatively cold dense target plasma may under certain circumstances affect the plasma equilibrium configuration. The source of this effect is non-potential part of the force acting on the plasma by atoms trapped in it. The equation for the shape of magnetic surfaces in open confinement systems is obtained. It is shown that appreciable distortion occurs even if the force is small as compared to the plasma pressure gradient.

Symmetrization of the atomic injection on azimuthal angle as the number of injectors increases efficiently promotes reduction of the magnetic surfaces distortions in axisymmetric open confinement systems. The dependence of the distortion upon the number of injectors is found and the role of fluctuations of injectors' parameters and their misplacement are elucidated. It is stressed that the infringement of axial symmetry of magnetic surfaces results in the occurrence of neoclassical transverse transport processes in the plasma.

1. Introduction

Energetic neutral injection into a dense target plasma is the principal approach being pursued for sustaining a hot plasma in magnetic fusion devices. When the neutrals are trapped by atomic processes their mechanical momentum is transferred to the plasma. The additional force $\boldsymbol{f}$ arising due to the absorption of the neutral atoms is very small in comparison with the pressure gradient; that is why it is usually neglected in the problem of plasma equilibrium though can play significant role in the problem of plasma stability as was first noted by D.D. Ryutov in 1983 [1].

If the force be really negligible, the effect of neutral injection on the plasma equilibrium is reduced to slow, in due course of plasma lifetime τ, reconstruction of the magnetic surfaces in accordance with the balance between trapping of the atoms into the plasma and losses of the plasma itself. It is of common knowledge [3, 4] that for such time scales a stationary solution may not exist since the plasma tends to spread out in radial direction. However on a shorter time interval the equilibrium still exists. In mirror confinement systems transition from almost arbitrary initial state of the plasma to equilibrium configuration takes the time of order $\Gamma^{-1} \ll \tau$ with Γ the growth rate of flute perturbations. We show below that the additional force $\boldsymbol{f}$ considerably affects the shape of equilibrium magnetic surfaces on the time scale $t \ll \tau$ if non-potential part $\boldsymbol{f}^t$ of the force is as small as p/R where p is the plasma pressure and R is the curvature radius of magnetic field line. Notice that similar condition $f \gtrsim p/R$ with the total additional force $\boldsymbol{f}$ instead of $\boldsymbol{f}^t$ assures that the effect of injection on flute stability prevails over the curvature effect [2].

If the magnetic field and injection system were axisymmetric by 100% (and, hence, the target plasma too) then the effect of the injection on the plasma equilibrium would be negligible since $f^t = 0$. However in real situation the injection can only approximately be considered as strictly axisymmetric since total symmetry formally requires infinite number of injectors to be used. Thus, accounting for real geometry of neutral injection results in violation of initial axial symmetry of the target plasma equilibrium which gives rise to neoclassical transport (see [5]).

2. Plasma equilibrium in mirror confinement systems

To find the plasma equilibrium we start with the pressure tensor

$$p_{\alpha\beta} = p_\perp \delta_{\alpha\beta} + (p_\parallel - p_\perp)\tau_\alpha \tau_\beta \tag{1}$$

where $\boldsymbol{\tau} = \boldsymbol{B}/B$ is the unit vector along a field line, and $p_\perp$ and $p_\parallel$ are the perpendicular and longitudinal pressure. Assuming zero plasma velocity we obtain

$$\frac{\partial p_{\alpha\beta}}{\partial x_\beta} = \frac{1}{c}[\boldsymbol{j}, \boldsymbol{B}]_\alpha + f_\alpha \,. \tag{2}$$

Projecting (2) onto the direction of magnetic field gives

$$\frac{\partial p_\parallel}{\partial s} = \frac{p_\parallel - p_\perp}{B}\frac{\partial B}{\partial s} + f_\parallel \tag{3}$$

with s being the co-ordinate along the field line. The perpendicular projection yields perpendicular component of the plasma current

$$\boldsymbol{j}_\perp = \frac{c}{B^2}\left[\boldsymbol{B}, \nabla p_\perp - \boldsymbol{f} + (p_\parallel - p_\perp)\boldsymbol{\kappa}\right] \tag{4}$$

where $\boldsymbol{\kappa}$ is the magnetic field curvature. Together with

$$\operatorname{curl} \boldsymbol{B} = \frac{4\pi}{c}\boldsymbol{j} \tag{5}$$

it gives the equation of transverse equilibrium

$$\nabla_\perp \left[p_\perp + \frac{B^2}{8\pi}\right] = \left[p_\perp - p_\parallel + \frac{B^2}{4\pi}\right]\boldsymbol{\kappa} + \boldsymbol{f}_\perp \,. \tag{6}$$

Longitudinal current $\boldsymbol{j}_\parallel$ is to be found from the continuity equation

$$\operatorname{div} \boldsymbol{j}_\perp + \operatorname{div} \boldsymbol{j}_\parallel = 0 \,. \tag{7}$$

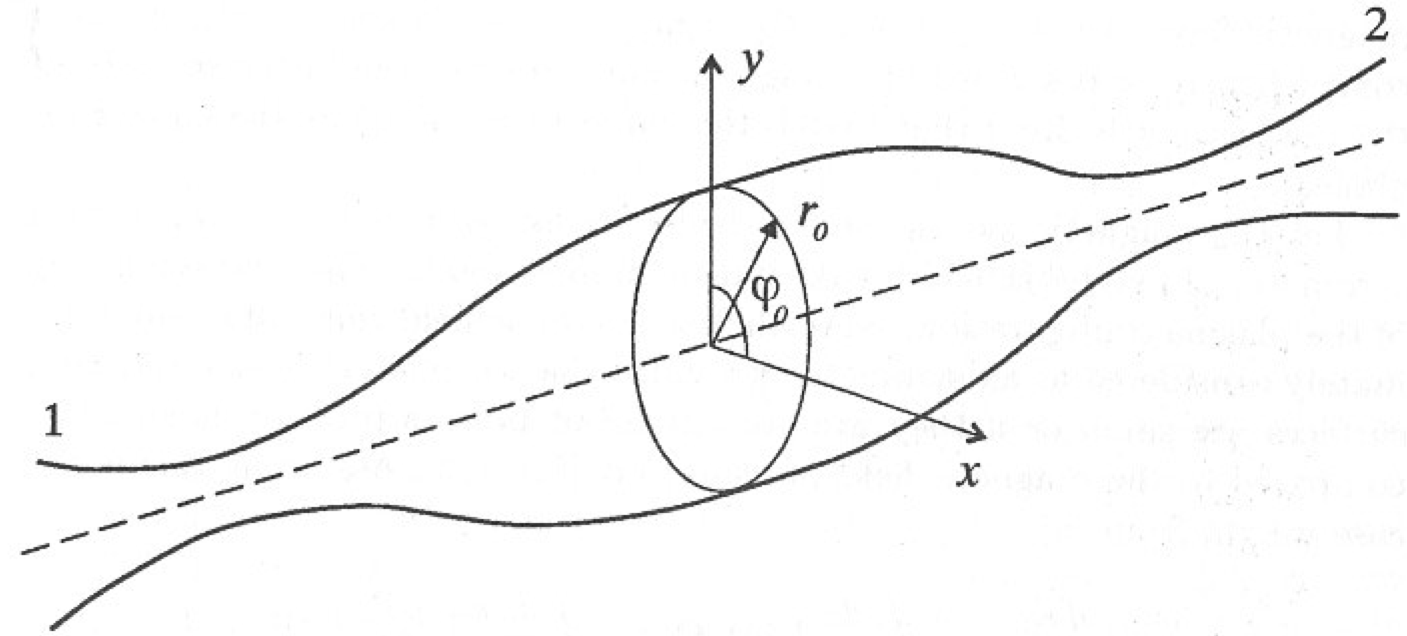

Figure 1: Sketch of mirror confinement system

Applying a routine algebra to the equations (7) and (4) as described in Ref. [5] we obtain the equation

$$\int \frac{ds}{B^2}[\boldsymbol{\tau}, \boldsymbol{\kappa}] \cdot \nabla(p_\perp + p_\parallel) = \int \frac{ds}{B^2}\{2[\boldsymbol{\tau}, \boldsymbol{\kappa}] \cdot \boldsymbol{f} - [\nabla, \boldsymbol{f}] \cdot \boldsymbol{\tau}\} \tag{8}$$

where the integration goes along field line between some points 1 and 2 (outside central cell) near which plasma density drops to a value small enough to consider that the plasma is detached from any conducting surfaces, see Fig. 1. Normally, plasma radius a is very small as compared to the radius of curvature $1/\kappa$ while $f \ll p_\perp + p_\parallel$ always. Hence, the first term in the right-hand-side of (8) can be neglected as well as the very last term in (3). Thus, we have

$$\int \frac{ds\, \kappa_n}{B^2}\frac{\partial}{\partial b}(p_\perp + p_\parallel) = -\int \frac{ds}{B^2}[\nabla, \boldsymbol{f}]_\tau \,. \tag{9}$$

The symbols n, b, and τ denote vector's components along the normal $\boldsymbol{n}$, binormal $\boldsymbol{b}$, and $\boldsymbol{\tau}$ with $\boldsymbol{\kappa} = \kappa_n \boldsymbol{n}$ and $\boldsymbol{b} = [\boldsymbol{\tau}, \boldsymbol{n}]$.

We introduce the polar coordinates r_0, φ_0 in a plane perpendicular to the device axis z. This plane can usually be thought of as an equatorial plane of mirror confinement system as shown on Fig. 1. Since the longitudinal force can be neglected in the equation (3) the usual isorrhopicity hypothesis takes place [7, 5]. It means that $p_\perp$ can be expressed as a function of $p_\parallel$ (or vice versa) and both are functions of the magnetic field strength B and of a magnetic flux. Hence, $p_\perp + p_\parallel = \mathcal{P}(r_0(\varphi_0), B)$. The dependence r_0 on φ_0 is comprehensively describes any magnetic surface since the latter consists of the magnetic field lines that match the curve $r_0 = r_0(\varphi_0)$ at the equatorial plane.

Let the magnetic system of the device be axisymmetric. However axial symmetry of magnetic field breaks if neutral injection breaks axial symmetry of the plasma configuration. Nevertheless magnetic field can still be approximately considered as axisymmetric provided that the distortions of magnetic surfaces are small or if they are not small but plasma pressure is small as compared to the magnetic field pressure, i.e. if $\beta \ll 1$. Assuming this is the case we get from (9)

$$\frac{d\,r_0}{d\,\varphi} = -\int \frac{ds}{B^2}\,[\mathrm{curl}\,\boldsymbol{f}]_\tau \Big/ \int \frac{ds\,\kappa_n}{rB^2}\,\frac{\partial \mathcal{P}}{\partial r_0}\,, \tag{10}$$

here and henceforth $\varphi_0 = \varphi$. Since r_0 is periodic function of φ we should require that

$$\int_0^{2\pi} d\varphi \int \frac{ds}{B^2}\,[\mathrm{curl}\,\boldsymbol{f}]_\tau = 0 \tag{11}$$

for the equation (10) to have a single-valued solution. It can be seen from what follows that this requirement determines the velocity of plasma rotation but the latter is zero for natural symmetry of neutral injection system (14).

3. Force, acting on the plasma

Before to calculate the force $\boldsymbol{f}$ we first try to explain why it can have solenoidal part such that $\mathrm{curl}\,\boldsymbol{f} \neq 0$.

Imagine that narrow beam of atoms moves along the axis y and enters the plasma by the distance x of the axis z directed along the plasma core. Then the force $\boldsymbol{f}$ has mainly the y component and suffers most rapid changes in the direction of the x axis, therefore $(\mathrm{curl}\,\boldsymbol{f})_z = \partial f_y/\partial x \neq 0$. If there exist one more beam, moving in the opposite direction along the same line, the force from the first beam is compensated but only partially due to attenuation of the beams inside the plasma core.

Another "philosophic question" is whether this force acts on plasma or on magnetic field. Some grounds for this question to be asked comes from the fact that impetus of particles gyrating in external magnetic field is not conserved. Speaking philosophically, we should agree that neutral atoms bring their momentum not only to the plasma but to the magnetic field (i.e.

to magnetic coils) too. However this effect has been in fact included into the equation (2) since the first term in the right-hand-side of the equation is nothing else as the divergency of Maxwell stress tensor $\sigma_{\alpha\beta} = (B^2/4\pi)\tau_\alpha\tau_\beta - (B^2/8\pi)\,\delta_{\alpha\beta}$. When being moved to the left-hand-side, this term together with $\partial p_{\alpha\beta}/\partial x_\beta$ forms divergency of total stress tensor of the system "plasma plus magnetic" field which is thus equal to the external force.

To calculate this force we adopt simple model of Ref. [2] for the motion of atom beams. We take into account only local trapping of atoms from original stream due to ionization and charge exchange and neglect a possibility of migration of the secondary atoms, which appear due to charge exchange to other points in the plasma or their departure from the plasma. Such a model is valid, in particular, when the velocity of secondary atoms is relatively low so that they are ionized not very far from the point where they originate in the plasma. It is also true under the conditions when the injection energy is so large that the trapping through the ionization goes much faster than the trapping through charge exchange but, on the other hand the injection energy is so small that the finite Larmor radius effects are negligible.

We shall assume the injected atoms to be monoenergetic and we shall denote their velocity by $\boldsymbol{v}_0$. For the spatial attenuation coefficient of the fast atoms we can then write down the following formula:

$$\varkappa = (n/v_0)(\langle\sigma_e v_e\rangle + \langle\sigma_i|\boldsymbol{v}_0 - \boldsymbol{v}_i|\rangle + \langle\sigma_{cx}|\boldsymbol{v}_0 - \boldsymbol{v}_i|\rangle) \tag{12}$$

where σ_e and σ_i are the cross sections for the ionization by electrons or by ions and σ_{cx} is the cross section for charge exchange by ions. The averaging is over the distribution functions of the plasma particles.

To describe injected beams we distinguish in the stream of atoms those which move in the direction given by the azimuthal angle χ (counted from the x axis in the xy plane) and by the polar angle θ (counted from the axis z) as shown on Fig. 2. This stream will be referred henceforth as partial stream. It can be thought of as produced by an injector with very narrow angular spread.

We introduce the Cartesian coordinates (ξ, η, ζ) connected with this partial stream. Let the axis ζ be directed along the "axis" of this beam, the axis η be perpendicular to the axis ζ of the beam as well as to the axis z of the plasma body, the axis ξ be perpendicular to η and ζ.

Imagine for a moment that the reference points of the two coordinate system coincide, i.e. the axis ζ of the beam intersects the axis of the plasma at the point $z = 0$. Then the two coordinate systems are related by the

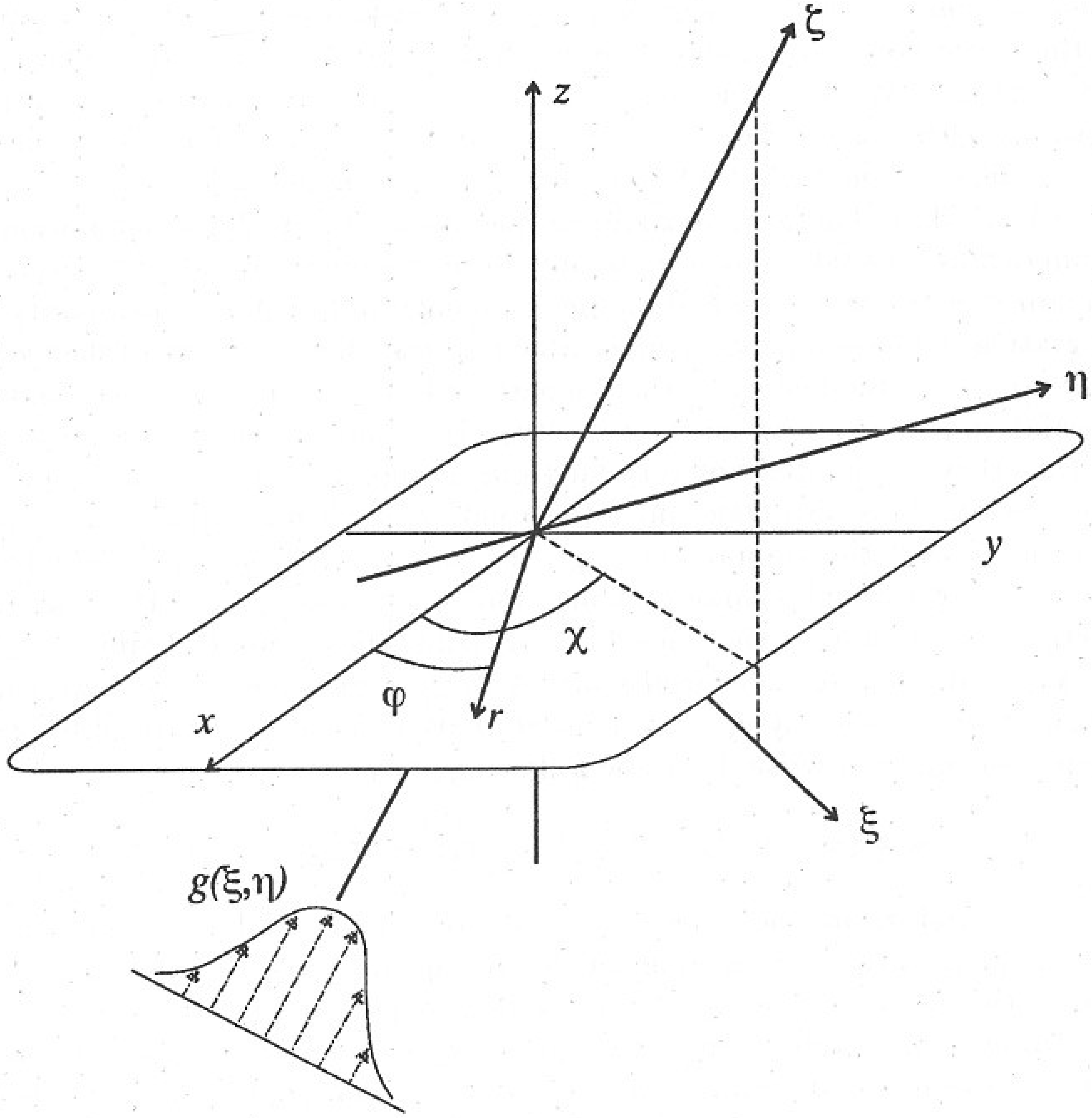

Figure 2: Space distribution of plane-parallel stream, moving along the axis ζ is characterized by the function $g(\xi,\eta)$. The axis z of plasma lies in the plane $\xi\zeta$; the axis η lies in the plane xy; the angles χ and φ are counted from the axis x in the xy plane. The reference point of the Cartesian systems of coordinates (ξ,η,ζ) connected with the nth beam has the coordinates $(0,0,z_n)$ in the Cartesian system of coordinates (x,y,z) connected with the plasma.

equations

$$\begin{aligned}
\xi &= x\cos\theta\cos\chi + y\cos\theta\sin\chi - z\sin\theta\,,\\
\eta &= -x\sin\chi + y\cos\chi\,,\\
\zeta &= x\sin\theta\cos\chi + y\sin\theta\sin\chi + z\cos\theta\,.
\end{aligned}$$

For given angles χ and θ, spatial distribution of the injected atoms across a partial stream is comprehensively described by the function $g(\xi,\eta)$. To write down the distribution function $F(\boldsymbol{r},\boldsymbol{v})$ of all injected atoms we first note that angular distribution of a neutral beam produced by a single injector is usually very narrow and can be approximated by delta functions $\delta(\theta-\theta_n)\,\delta(\chi-\chi_n)$ with θ_n and χ_n indicating angle position of the nth injector. Our second point is that spatial distribution $g(\xi,\eta)$ across any of the beams is one and the same for all injectors; we normalize it to unity:

$$\iint d\xi\, d\eta\, g(\xi,\eta) = 1\,.$$

Significant variations can only be observed in currents of beams $\mathcal{I}_n$ produced by different injectors [6]. Third point is that the injectors might be directed to different points z_n on the axis of the plasma. This possibility can be taken into account if we write down spatial distribution as $g(\xi + z_n\sin\theta,\eta)$ instead of $g(\xi,\eta)$. Thus we have

$$F(\boldsymbol{r},\boldsymbol{v}) = \sum_{n=0}^{N-1} \mathcal{I}_n g(\xi + z_n\sin\theta,\eta)\delta(\theta-\theta_n)\,\delta(\chi-\chi_n)\,\delta(v-v_n)/v^3\sin\theta \quad (13)$$

where the summation goes over all of N injectors. In agreement with the conditions normally encountered we shall assume any of the partial beams to have special symmetry so that the function g is even on its second argument:

$$g(\xi,-\eta) = g(\xi,\eta)\,. \quad (14)$$

With this symmetry being taken into account, the requirement (11) is satisfied if velocity of plasma rotation is zero. As a rule, the function g is also even on its first argument but it is not so important for what follows.

To calculate the density n_b of primary atoms at a point (x,y,z) the attenuation factor

$$\exp\left[-\int_{-\infty}^{0} d\zeta'\, \varkappa(x+\zeta'\sin\theta_n\cos\chi_n, y+\zeta'\sin\theta_n\sin\chi_n, z+\zeta'\cos\theta_n)\right]$$

for every partial stream has to be included. Then

$$\begin{aligned} n_b(\mathbf{r}) &= \int d^3v\, F(\mathbf{v},\mathbf{r}) \exp\left[-\int_{-\infty}^{0} d\zeta'\, \varkappa\right] \\ &= \sum_{n=0}^{N-1} (\mathcal{I}_n/v_n) g[r\cos\theta_n \cos\alpha_n - (z-z_n)\sin\theta_n, r\sin\alpha_n] \\ &\quad \times \exp\left[-\int_{-\infty}^{0} d\zeta'\, \varkappa\right], \end{aligned} \tag{15}$$

where $\alpha_n = \varphi - \chi_n$, φ is the azimuthal angle in the cylindrical system of coordinates (r, φ, z), see Fig. 2.

Single neutral beam acts on unit plasma volume with the force

$$f_n = M v_n \mathcal{I}_n \varkappa g[r\cos\theta_n \cos\alpha_n - (z-z_n)\sin\theta_n, r\sin\alpha_n] \exp\left[-\int_{-\infty}^{0} d\zeta'\, \varkappa\right]$$

directed along the velocity $\mathbf{v}_n$. The radial, azimuthal and the longitudinal components of the total force produced by all beams are, respectively, equal to

$$(f_r, f_\varphi, f_z) = \sum_{n=0}^{N-1} (\sin\theta_n \cos\alpha_n, -\sin\theta_n \sin\alpha_n, \cos\theta_n) f_n\,. \tag{16}$$

4. Magnetic surfaces

To get an idea of how large can the distortion of the magnetic surfaces be under the action of the neutral injection we consider the model of a mirror device shown on Fig. 1 and estimate the force exerted on the plasma by single neutral beam.

Typically, mirror system consists of a section of a uniform magnetic field of length L (where the injection is placed) and two mirror sections of length $L_{\rm m}$; the radius of the system is small as compared to $L_{\rm m}$. For evaluation purpose we assume that attenuation length of the beam is optimal, i.e., comparable with the plasma radius, $\varkappa \sim a^{-1}$. For a larger attenuation length the plasma would become transparent to the beam and the efficiency of the injection would drop. For a smaller attenuation length the plasma would become hollow.

We denote the width of the beam in the direction of plasma axis by $L_{\rm inj}$; we shall see that it does not enter final estimation. As to the beam width D in direction across the plasma it is usually of order plasma radius, $D \sim a$.

Injection usually goes at not very slope angle to the system axis so that $\theta \sim 1$. With these assumptions adopted, the total number of hot ions produced in the plasma per unit of time can be estimated just as $\mathcal{I}_n$.

Lifetime of fast ions is determined by electron drag (here and henceforth only ions with energy above a substantial part, say one third, of initial energy of neutral atoms are referred as fast ions). It is slowing down on relatively cold target plasma electrons but not the losses along magnetic field that restricts pressure of hot plasma component. Since fast ions occupy almost total volume $\pi a^2 L$ of the mirror device their density n_h is to be estimated from the following material balance relation

$$I_n \sim a^2 L n_{\rm h}/\tau_{\rm drag}$$

where $\tau_{\rm drag}$ is the electron drag time. Using this relation we can express the force $f_n \sim \varkappa m v_n \mathcal{I}_n / D L_{\rm inj}$ acting on the plasma from the side of single neutral beam through the plasma pressure $p_{\rm h} \sim M v_n^2 n_{\rm h}$:

$$f_n \sim (L/L_{\rm inj})(p_{\rm h}/v_n \tau_{\rm drag})\,.$$

Bearing in mind that the curvature of a field line is of order $a/L_{\rm m}^2$, we evaluate from (10) the magnitude of radial distortion of the magnetic surface with mean radius (in the equatorial plane) a:

$$\frac{\Delta r}{a} \sim \frac{L L_{\rm m}}{v_n \tau_{\rm drag} a} \frac{p_{\rm h}}{p_{\rm c} + p_{\rm h}}$$

where $p_{\rm c}$ is the pressure of cold target plasma. For the following set of parameters $a = 10$cm, $L = 2L_{\rm m} = 10^3$cm, $v_n = 10^8$cm/s, $\tau_{\rm drag} = 1 \div 10$msec, $p_{\rm h} \gg p_{\rm c}$, relevant to Gas-Dynamic Trap [8], it yields $\Delta r/a \sim 1 \div 0.1$.

Radial distortions of magnetic surfaces can be significantly reduced by careful design of the injector system. To confirm this obvious prediction we consider a set of N equal injectors equidistantly placed on azimuthal angle, and equally inclined to the device axis z. In particular, we adopt that

$$\mathcal{I}_n = \mathcal{I}_0\,, \quad \chi_n = 2\pi n/N\,, \quad \theta_n = \theta_0\,, \quad v_n = v_0\,. \tag{17}$$

Since inhomogeneity of plasma and magnetic field in the region of injection is usually small we adopt that neutral beams enter the target plasma at the section of uniform field; it allows us from the very beginning to discard the exact coordinate z_n of the point where nth beam intersects the plasma axis since z_n does not enter final result of our calculations. Substituting summation over n in (16) by integration over χ by means of

$$\sum_{n=0}^{N-1} \mathcal{I}_n \to \int_0^{2\pi} d\chi\, I(\chi)$$

with $I(\chi)$ defined by the relation

$$I(\chi) = \sum_{n=0}^{N-1} \mathcal{I}_0 \delta(\chi - 2\pi n/N) , \tag{18}$$

we get

$$\begin{aligned}(\mathrm{curl}\, \boldsymbol{f})_z = &-M v_0 \sin\theta_0 \int_0^{2\pi} d\chi\, I(\chi) \\ &\times \left[\frac{1}{r}\frac{\partial}{\partial r} r \sin(\varphi - \chi) + \frac{1}{r}\frac{\partial}{\partial \varphi} \cos(\varphi - \chi)\right] \\ &\times \varkappa(r,\varphi)\, g(r\cos(\varphi-\chi)\cos\theta_0 - z\sin\theta_0, r\sin(\varphi-\chi)) \\ &\times \exp\left[-\int_{-\infty}^{0} d\zeta'\, \varkappa\left(\sqrt{r^2 + 2r\zeta'\sin\theta_0\cos(\varphi-\chi) + \zeta'^2\sin^2\theta_0}\right)\right] \end{aligned} \tag{19}$$

Here both $\partial/\partial r$ and $\partial/\partial\varphi$ act on all of multiples to the right of them; the same agreement is true for the integration sign. Taking into account intrinsic symmetry of the neutral beams (14) it is easy to show that $(\mathrm{curl}\, \boldsymbol{f})_z$ tends to zero as $N \to \infty$ providing the target plasma to be axisymmetric, i.e. $\varkappa$ does not depend on the azimuthal angle φ. We adopt below that $\varkappa = \varkappa(r)$ which is true if the distortion of magnetic surfaces due to atomic injection is small enough.

Note also that the first term $r\cos(\varphi - \chi)\cos\theta$ in the first argument of the function g in (19) may be omitted right now since keeping it does not change the result of integration over magnetic field length (i.e. over z) in the equations (9), (10), or (11).

To find asymptotic behavior of $(\mathrm{curl}\, \boldsymbol{f})_z$ as $N \to \infty$ we introduce new integration variable $\alpha = \varphi - \chi$ instead of χ and we note that $\partial/\partial\varphi = -\partial/\partial\alpha$. Next step is to expand $I(\chi)$ into Fourier series. Inserting

$$I(\chi) = \sum_{k=-\infty}^{\infty} I_k \exp(ik\varphi) , \tag{20}$$

into (19) and integrating by parts the term with $\partial/\partial\varphi$ we get

$$\begin{aligned}(\mathrm{curl}\, \boldsymbol{f})_z \;=\; & 2\pi M v_0 \sin\theta_0 \sum_{k=-\infty}^{\infty} \left[r^{-k-1}\frac{d}{dr} r^{k+1}\right] \\ & \times \varkappa(r)\left[I_k J_{k+1}\, \mathrm{e}^{ik\varphi} - I_{-k} J_{-k-1}\, \mathrm{e}^{-ik\varphi}\right]/2i \end{aligned} \tag{21}$$

where

$$I_k = \int_0^{2\pi} \frac{d\chi}{2\pi} I(\chi)\, \mathrm{e}^{-ik\chi} , \tag{22}$$

$$\begin{aligned} J_k \;=\; & \int_0^{2\pi} \frac{d\alpha}{2\pi} g(-z\sin\theta, r\sin\alpha)\, \mathrm{e}^{-ik\alpha} \\ & \times \exp\left[\int_{-\infty}^{r\cos\alpha} \frac{d\zeta'}{\sin\theta}\, \varkappa\left(\sqrt{r^2\sin^2\alpha + \zeta'^2}\right)\right] . \end{aligned} \tag{23}$$

It is easy to see from (18) and (22) that

$$I_k = \mathcal{I}_0 \begin{cases} N/2\pi & \text{if } k = mN \\ 0 & \text{if } k \neq mN \end{cases} \tag{24}$$

where m is integer. This means that summation in (21) goes over integer multiples m of N. The $m = 0$ term turns out to be zero due to (14) so that the most significant terms come from $m = \pm 1$. Other terms diminish rapidly as m grows provided that $N \gg 1$. Thus, we can restrict ourselves to asymptotic calculation of the integral (23) with $k \to \infty$. The asymptote to be calculated works well at moderate number of injectors, $N = 4 \div 6$, which is the case for major fusion devices.

Assuming diffuse profiles for both beam spatial distribution,

$$g(\xi, \eta) = \frac{1}{\pi L_{\mathrm{inj}} D} \exp\left(-\frac{\xi^2}{L_{\mathrm{inj}}^2} - \frac{\eta^2}{D^2}\right) ,$$

and the coefficient of neutral beam attenuation in plasma,

$$\varkappa = \varkappa_0 \exp(-r^2/a^2) ,$$

for the case $N \gg 1$ and $\varkappa_0 a/sin\theta_0 \gg (a/D)^2 \exp\left(-|k|a^2/2D^2\right)$ we get

$$\begin{aligned}\int \frac{ds}{B^2} [\nabla, \boldsymbol{f}]_r = & \frac{\varkappa_0 M v_0 \mathcal{I}_0 N \sin\theta_0}{2\pi D B_{\mathrm{min}}^2} \sum_{m\neq 0} (-1)^{mN+1} r^{-mN-1} \frac{d}{dr} r^{mN+1} \\ & \times \left[\frac{\mathrm{e}r^2}{2D^2}\right]^{|mN+1|/2} |mN+1|^{-|mN|/2-1} \exp\left(-\frac{r^2}{D^2}\right) \sin mN\varphi \end{aligned} \tag{25}$$

where $B_{\rm min}$ is the magnetic field in the uniform section. Details of calculations can be found in Appendix A.

Putting $\varkappa_0 \sim a^{-1}$ we find for the amplitude Δr of the distortion of magnetic surface with the mean radius a the following estimate:

$$\frac{\Delta r}{a} \sim \frac{LL_{\rm m}}{v_0 \tau_{\rm drag} a} \frac{p_{\rm h}}{p_{\rm c} + p_{\rm h}} N^{-N/2} .$$

Indeed, it decreases rapidly as N increases.

5. The effect of injectors nonidentity

In previous section all injectors are assumed to be identical. They had equal currents $\mathcal{I}_0$, were placed equidistantly on azimuthal angle and had one and the same spatial distribution $g(\xi, \eta)$.

It is easy to discard first and second of these assumptions. The case where spatial distribution function varies from injector to injector will be considered in companion paper [9]. In this section we take into account that in real experiments the injectors produce neutral beams with different currents $\mathcal{I}_n$ and may be misplaced on azimuthal angle from equidistant positions. Just to elucidate the effect of such variations on plasma equilibrium we assume them to be occasional though statistical approach may not be as good if the number of injectors is moderate.

Let

$$\mathcal{I}_n = \sigma_n \mathcal{I}_0 = (1 + \delta\sigma_n)\mathcal{I}_0 , \quad \chi_n = 2\pi n/N + \delta\chi_n ,$$

with statistical average of $\delta\sigma_n$ and $\delta\chi_n$ being zero:

$$\langle \delta\sigma_n \rangle = 0 , \quad \langle \delta\chi_n \rangle = 0 .$$

We assume statistical properties of σ_n and $\delta\chi_n$ to be equal for all injectors; therefore the index n will be omitted below in averaged quantities.

To find $\langle (\mathrm{curl}\, \boldsymbol{f})_z \rangle$, the Fourier amplitude I_k in the formula (21) of previous section should be substituted by its averaged value

$$\langle I_k \rangle = \mathcal{I}_0 \left\langle \sigma\, \mathrm{e}^{-ik\delta\chi} \right\rangle \begin{cases} N/2\pi & \text{if } k = mN \\ 0 & \text{if } k \neq mN \end{cases} \tag{26}$$

instead of (24) calculated for the set (17) of equidistantly spaced identical injectors. Since $\langle I_k \rangle = 0$ for $k \neq mN$, cancellation of lower azimuthal distortions of magnetic surfaces still takes place, however it is not as complete as in previous section because $\langle I_k^2 \rangle \neq 0$. If variations of injectors' parameters are not correlated we obtain

$$\left\langle |\delta I_k|^2 \right\rangle = N\mathcal{I}_0^2 \left\{ \langle \sigma^2 \rangle - |\langle \sigma\, \mathrm{e}^{-ik\delta\chi} \rangle|^2 \right\} / (2\pi)^2 \tag{27}$$

where δI_k denotes $I_k - \langle I_k \rangle$.

If, in particular, there is no azimuthal misplacement, $\delta\chi = 0$, then

$$\left\langle |\delta I_k|^2 \right\rangle = N\mathcal{I}_0^2 \langle \delta\sigma^2 \rangle / (2\pi)^2 .$$

Fluctuations of the injectors' currents are negligible provided that additional distortions induced by them are much smaller than the amplitude of the largest distortion with $k = N$, i.e. $\sqrt{\langle \delta\sigma^2 \rangle} \ll N^{-(N+1)/2}$.

If, on the contrary, the currents are equal, i.e. $\delta\sigma = 0$, and uncontrolled azimuthal misplacement of injectors prevail, then

$$\left\langle |\delta I_k|^2 \right\rangle = N\mathcal{I}_0^2 \left(1 - \langle \cos k\delta\chi \rangle^2\right) / (2\pi)^2 .$$

The condition for this effect to be neglected is $\sqrt{\langle \delta\chi^2 \rangle} \ll N^{-(N+3)/2}$.

A. Calculation of J_k

To calculate the integral

$$\tilde{J}_k \equiv \int dz\, J_k = \frac{1}{\sqrt{\pi} D} \int_0^{2\pi} \frac{d\alpha}{2\pi}\, \mathrm{e}^S \tag{A1}$$

where

$$\begin{aligned} S \;&=\; -ik\alpha - (r^2/D^2)\sin^2\alpha \\ &\quad -\varkappa_0 \int_{-\infty}^{r\cos\alpha} \frac{d\zeta'}{\sin\theta_0} \exp\left[-\frac{r^2\sin^2\alpha + \zeta'^2}{a^2}\right] \end{aligned} \tag{A2}$$

we transform the contour of integration in the complex plane $\alpha = \mu + i\nu$ so that it should pass through the stationary points to be found from the equation

$$\frac{dS}{d\alpha} = 0 .$$

For the sake of simplicity we first neglect the last term in (A2) which is proportional to $\varkappa$; it describes attenuation of neutral beam in the plasma. The coordinates of stationary points are then equal to

$$\mu = \pi l/2 , \quad k = -(-1)^l (r/D)^2 \sinh^2\nu \tag{A3}$$

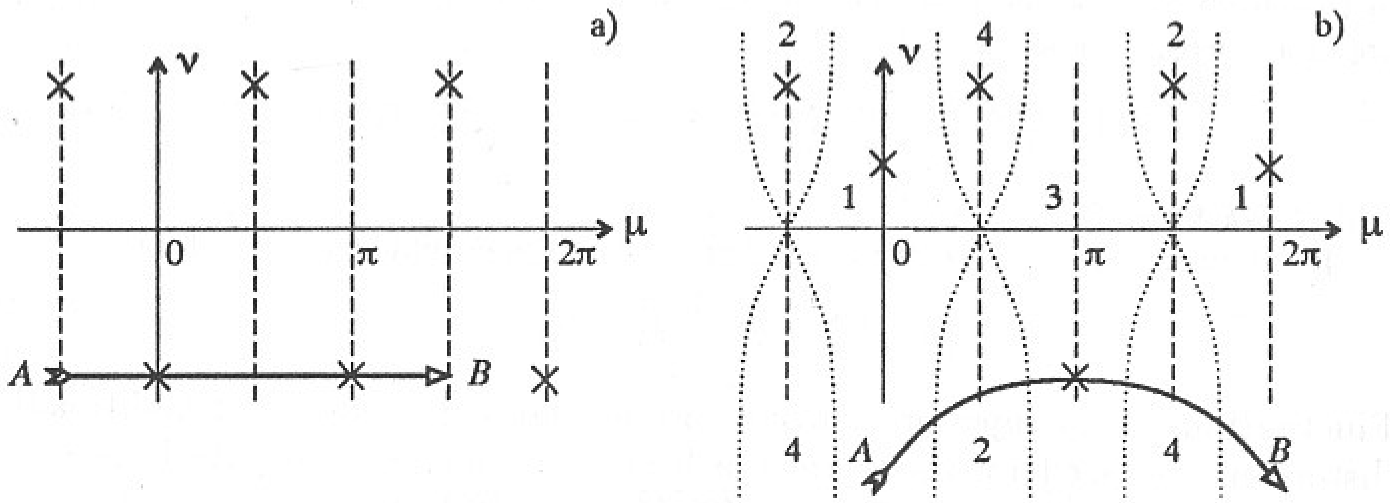

Figure 3: To calculation of $\tilde{J}_k$. Stationary points are marked by crosses. Transformed contour of integration goes from the point A to the point B: $a)$—$\varkappa_0 a/\sin\theta_0 \ll (a/D)^2\exp(-|k|a^2/2D^2)$; $b)$—$\varkappa_0 a/\sin\theta_0 \gg (a/D)^2\exp(-|k|a^2/2D^2)$. Dotted lines show the boundaries $\arg\cos\alpha = \pm\pi/4$ between the regions of different asymptotic behavior of the function G as given by (A6). The numbers *1*–*4* corresponds to the line number in the equation (A6).

with l being an integer. Complex α-plane is shown on Fig. 3a for the case $k > 0$ (to get the opposite case $k < 0$ it is sufficient to inverse the sign of ν). Odd values of l correspond to stationary points in upper half of complex α-plane while points with even l are in lower half.

Transformed contour of integration should have such a shape that local maxima of real part of S would be placed at the stationary points the contour matches. One of the possible countours which conforms to this requirement is shown on Fig. 3a. It is parallel to the axis μ (initial and final ends of the contour can be moved from the axis μ by equal distances since S is periodic function of μ).

Second derivative of S:

$$\frac{d^2S}{d\alpha^2} = -(-1)^l\sqrt{(r/D)^4 + k^2}$$

is negative at the stationary points $l = 0$ and $l = 2$ situated at the transformed contour in the lower half-plane of complex α. Hence, the function S at the contour of integration near these points can be expanded as

$$S = -i\pi kl/2 + k\nu - \frac{r^2}{D^2}\sinh^2\nu - \left|\frac{d^2S}{d\alpha^2}\right|\frac{(\mu - \pi kl/2)^2}{2}\,.$$

Performing integration with this expansion inserted into (A1) we find

$$\tilde{J}_k = \frac{1}{2\pi D}\frac{[1+(-1)^k]}{\sqrt[4]{r^4/D^4+k^2}}\exp\left[\sqrt{r^4/D^4+k^2}/2 - (r/D)^2/2\right] \times \left[\frac{r^2/D^2}{\sqrt{r^4/D^4+k^2}+|k|}\right]^{|k|/2}. \qquad \text{(A4)}$$

Note that $\tilde{J}_k = 0$ if k is *odd*. This means that for small trapping coefficient $\varkappa_0$ the first nonzero terms $k = \pm N$ in the series (21) are as small as $\varkappa_0^2$ if the number of injectors is *even*.

Let now proceed to the case $\varkappa \neq 0$ and take into account the last term in (A2):

$$-(\varkappa_0 a/\sin\theta_0)\exp\left(r^2/a^2\right)G[(r/a)\cos\alpha] \qquad \text{(A5)}$$

where

$$G(t) = e^{t^2}\int_{-\infty}^{t} dt'\, e^{-t'^2}\,.$$

Asymptotic behavior of the function $G(t)$ as $|t| \to \infty$ depends on the quadrants which its argument belongs to:

$$G(t) \approx \begin{cases} \sqrt{\pi}\,e^{t^2} - \frac{1}{2t} & \text{if } -\frac{pi}{4} < \arg t < \frac{\pi}{4}, \\ \frac{\sqrt{\pi}}{2}\,e^{t^2} - \frac{1}{2t} & \text{if } \frac{\pi}{4} < \arg t < \frac{3\pi}{4}, \\ -\frac{1}{2t} & \text{if } \frac{3\pi}{4} < \arg t < \frac{5\pi}{4}, \\ \frac{\sqrt{\pi}}{2}\,e^{t^2} - \frac{1}{2t} & \text{if } \frac{5\pi}{4} < \arg t < \frac{7\pi}{4}. \end{cases} \qquad \text{(A6)}$$

The boundaries $\arg t = \pm\pi/4$ of the quadrants after being projected onto α-plane is given by the equation $\tanh\nu = \pm\cot\mu$. They are shown on Fig. 3b by dotted lines; the numbers from *1* to *4* consecutively mark the regions from $-\pi/4 < \arg t < \pi/4$ to $5\pi/4 < \arg t < 7\pi/4$.

Using asymptotic formula (A6) one can show that the stationary points $l = 1, 2, 3$ (i.e. $\mu = \pi/2, \pi, 3\pi/2$) stay almost unmoved as $\varkappa_0$ increases since they are placed in the regions *4*, *3*, *2* respectively where the last term in (A2) is small. The ν-coordinate of these points can approximately be determined from the equation (A3).

The point $l = 0$ moves along the vertical line $\mu = 0$ upwards (remember that $k > 0$). Its ν-coordinate is to be determined from the equation

$$k = -\frac{r^2}{D^2}\sinh 2\nu + \frac{\sqrt{\pi}\varkappa_0 r^2}{a}\sinh 2\nu\exp\left(\frac{r^2}{a^2}\sinh^2\nu\right).$$

If

$$\varkappa_0 a > \frac{2}{\sqrt{\pi}} \frac{a^2}{D^2} \exp\left(-\frac{|k|a^2}{2D^2}\right) \sin\theta_0 \qquad (A7)$$

this point passes into the upper half-plane and the only stationary point left on the transformed contour of integration is $l = 2$. Hence, omitting the term in (A4) which comes from other point $l = 0$, i.e. the unity in the sum $[1 + (-1)^k]$ we find $\tilde{J}_k$ for the case when the inequality (A7) is satisfied with a certain margin. Expanding (A4) on small parameter r^2/kD^2 finally yields

$$\tilde{J}_k = \frac{(-1)^k}{2\pi D\sqrt{|k|}} \left|\frac{er^2}{2D^2 k}\right|^{|k|/2}. \qquad (A8)$$

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