41

45

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NUCLEAR STRUCTURE CORRECTIONS TO DEUTERIUM HYPERFINE STRUCTURE AND LAMB SHIFT

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НОВОСИБИРСК

Nuclear structure corrections to deuterium hyperfine structure and Lamb shift

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Abstract

The low-energy theorem for the forward Compton scattering is generalized to the case of an arbitrary target spin. The generalization is used to calculate the corresponding contribution to the deuterium hyperfine structure. The nuclear-structure corrections are quite essential in this case due to the deuteron large size. Corrections of this type calculated here remove the discrepancy between the theoretical and experimental values of the deuterium hyperfine splitting. Explicit analytical result is obtained also for the deuteron polarizability contribution to the Lamb shift.

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1 Introduction

The hyperfine (hf) splitting in deuterium ground state has been measured with high accuracy. The most precise experimental result for it was obtained with an atomic deuterium maser and constitutes [1]

$$\nu_{exp} = 327\,384.352\,522\,2(17)\,\text{kHz}. \tag{1}$$

Meanwhile the theoretical calculation including higher order pure QED corrections gives

$$\nu_{QED} = 327\,339.27(7)\,\text{kHz}. \tag{2}$$

The last number was obtained by using the theoretical result for the hydrogen hf splitting from Ref. [2]

$$1\,420\,451.95(14)\,\text{kHz}$$

which does not include proton structure and recoil radiative correction, and combining it with the theoretical ratio of the hf constants in hydrogen and deuterium from Ref. [3]

$$4.339\,387\,6(8)$$

based on the ratio of the nuclear magnetic moments and including the reduced mass effect in $|\psi(0)|^2$.

In the present article the discrepancy

$$\nu_{exp} - \nu_{QED} = 45\,\text{kHz} \tag{3}$$

is removed by taking into account the effects due to the finite size of deuteron. Such effects are obviously much larger in deuterium than in hydrogen. One nuclear-structure contribution to the deuterium hf splitting was pointed out long ago [4] by some intuitive arguments, and then discussed in more detail in Refs. [5, 6, 7]. Here we treat the deuteron finite-size effects in a systematic way. Not only the old result is reproduced, but new corrections are obtained, among them that generated by the deuteron electric and magnetic form-factors.

To calculate some contributions to the deuterium hf structure we generalize the low-energy theorem for the Compton scattering to an arbitrary target spin.

One more subject considered in this paper is the contribution of the deuteron polarizability to the deuterium Lamb shift. The fact that this contribution is close to the accuracy attained in experiment was pointed out in Refs. [8, 9] where the effect was calculated in the square-well approximation for the nuclear potential. This correction was calculated then with separable nuclear potentials in Ref. [10]. Here we obtain in the zero-range approximation a closed analytical result for the deuteron polarizability contribution to the Lamb shift.

2 Low-energy theorem for the forward Compton scattering on an arbitrary target

According to the well-known low-energy theorem for the Compton scattering on a spin 1/2 hadron [11, 12], this amplitude is described by the pole Feynman diagrams. We are interested here not in the spin-independent Thomson amplitude which is of zeroth order in the photon frequency ω, but in the next, spin-dependent term of the ω expansion. This contribution as well can be easily obtained directly in the nonrelativistic approximation [13]. In this approximation the electromagnetic vertex can be immediately written for an arbitrary spin $\vec{s}$:

$$\frac{e}{2m_p}\left\{\frac{Z}{A}(2\vec{p}+\vec{k})+\frac{\mu}{s}\,i[\vec{s}\times\vec{k}]\right\}. \tag{4}$$

Here Z is the hadron charge, and its g-factor is related as follows to the magnetic moment μ measured in the nuclear magnetons $e/2m_p$:

$$g=\mu/s.$$

In the forward scattering case, when the hadron is at rest ($\vec{p}=0$) and both photons have physical, transverse polarizations ($(\vec{k}\vec{e})=(\vec{k}\vec{e}\,')=0$), this vertex reduces to the pure spin interaction. The nonrelativistic pole scattering amplitude generated by this interaction is

$$M_1 = M_{1mn}e'_m e_n = \left(\frac{e}{2m_p}\right)^2 \omega\, g^2\, i(\vec{s}\cdot[\vec{e}\,'\times\vec{e}]). \tag{5}$$

However, this expression is incomplete. Indeed, being applied to a proton, it does not reproduce the well-known result [11, 12] according to which the spin-dependent forward scattering amplitude is proportional to $(g-2)^2$. The explanation was pointed out in Ref. [13]: the nonrelativistic pole amplitude should be supplemented by a contact term generated by the spin-orbit interaction, which restores the agreement with the classical result [11, 12].

This contact term can be easily derived for an arbitrary spin (as well as the nonrelativistic pole contribution (5)) in the following way. The motion equation for spin in an external electric field $\vec{E}$ can be written to lowest nonvanishing order in v/c as

$$\frac{d\vec{s}}{dt} = \frac{e}{2m_p}\left(g-\frac{Z}{A}\right)\left[\vec{s}\times[\vec{E}\times\vec{v}]\right]. \tag{6}$$

Here A is the target mass as measured in the units of m_p (i.e., it is the atomic number in the case of nuclei we are mainly interested in in the present work.) The interaction Hamiltonian generating equation (6) is obviously

$$H_s = -\frac{e}{2m_p}\left(g-\frac{Z}{A}\right)\left(\vec{s}\cdot[\vec{E}\times\vec{v}]\right). \tag{7}$$

Expressions (6), (7) are in fact only slightly rewritten formulae from book [14]. After substituting

$$\vec{v} = \frac{\vec{p}-Ze\vec{A}}{Am_p}$$

into Hamiltonian (7), we arrive at the following contact interaction:

$$V_c = \left(\frac{e}{2m_p}\right)^2 \frac{2Z}{A}\left(g-\frac{Z}{A}\right)\left(\vec{s}\cdot[\vec{E}\times\vec{A}]\right). \tag{8}$$

It produces one more piece in the scattering amlitude:

$$M_2 = M_{2mn}e'_m e_n = \left(\frac{e}{2m_p}\right)^2 \omega\left(-4\frac{Z}{A}\right)\left(g-\frac{Z}{A}\right) i(\vec{s}\cdot[\vec{e}\,'\times\vec{e}]). \tag{9}$$

Taken together, expressions (5) and (9) generate the forward scattering amplitude:

$$M = \left(\frac{e}{2m_p}\right)^2 \omega \left(g - 2\frac{Z}{A}\right)^2 i(\vec{s}\cdot[\vec{e}' \times \vec{e}]). \tag{10}$$

This is the generalization of the low-energy theorem we are looking for.

In the particular case of a proton ($s = 1/2$, $Z = A = 1$) this formula reduces to the result of Refs. [11, 12].

3 Low-energy theorem and deuterium hyperfine structure

Being dependent on nuclear spin, the low-energy amplitude obtained contributes to the atomic hyperfine structure. However, to apply it to this problem, the amplitude should be modified. Indeed, both photons exchanged between the nucleus and electron are off-mass-shell. So, here $\omega \neq |\vec{k}|$. Then, virtual photons have extra polarizations. We will use the gauge $A_0 = 0$ where the photon propagator is

$$D_{im}(\omega, \vec{k}) = \frac{d_{im}}{\omega^2 - \vec{k}^2}, \quad d_{im} = \delta_{im} - \frac{k_i k_m}{\omega^2}; \quad D_{00} = D_{0m} = 0. \tag{11}$$

Now, first of all, the magnetic moment contribution M_{1mn} to the pole diagram changes to

$$\tilde{M}_{1mn} = \left(\frac{e}{2m_p}\right)^2 g^2\, i\,\epsilon_{mnk} k_k (\vec{k}\cdot\vec{s})\,\frac{1}{\omega}. \tag{12}$$

Second, the convection current, which is proportional to $\pm\vec{k}$ for a nucleus at rest, is operative now and induces the following nuclear-spin-dependent contribution to the forward scattering amplitude:

$$M_{3mn} = -\left(\frac{e}{2m}\right)^2 2\frac{Z}{A}\, g\, i\,(k_m \epsilon_{nrs} k_r s_s - k_n \epsilon_{mrs} k_r s_s)\frac{1}{\omega}. \tag{13}$$

We are ready now to write down the electron-nucleus nuclear-spin-dependent scattering amplitude generated by the two-photon exchange with the deuteron intermediate state:

$$T_{el} = 4\pi\alpha i \int \frac{d^4k}{(2\pi)^4}\, \frac{d_{im} d_{jn}}{k^4}\, \frac{\gamma_i(\hat{l} - \hat{k} + m_e)\gamma_j}{k^2 - 2lk} (\tilde{M}_{1mn} + M_{2mn} + M_{3mn}). \tag{14}$$

Here $l_\mu = (m_e, 0, 0, 0)$ is the electron momentum. The structure $\gamma_i(\hat{l} - \hat{k} + m_e)\gamma_j$ reduces to $-i\,\omega\,\epsilon_{ijl}\sigma_l$ where $\vec{\sigma}$ is the electron spin. We will calculate this Feynman integral with logarithmic accuracy. Two subtle points should be mentioned here. The singularity at $\omega = 0$, originating from $1/\omega^2$ in the projection operator (11), should be understood in the sense of the principal value. Then, one comes across a term containing integral

$$\int_0 \frac{d|\vec{k}|}{|\vec{k}|^2}$$

which diverges in a power-like way at $|\vec{k}| \to 0$. To regularize it one should introduce non-vanishing electron velocity v; in this way one obtains the well-known Coulomb wave-function correction $\pi\alpha/v$ which should be neglected to our purpose. In our logarithmic approximation one should neglect as well possible constant terms originating from this integral.

The final result of calculation can be conveniently presented in the following form. Let us write down the spin-dependent Born term in the electron-nucleus scattering amplitude:

$$T_0 = -\frac{2\pi\alpha}{3m_e m_p}\, g\,(\vec{\sigma}\cdot\vec{s}). \tag{15}$$

It is in fact minus Fourier-transform of the lowest order contact hyperfine interaction. Therefore, the ratio $\Delta_{el} = T_{el}/T_0$ is nothing else but the relative magnitude of the discussed correction to the hf structure. The result for integral (14) can be written as

$$\Delta_{el} = \frac{3\alpha}{8\pi}\frac{m_e}{m_p} \log\frac{\Lambda}{m_e}\frac{1}{g}\left(g^2 - 4g\frac{Z}{A} - 12\frac{Z^2}{A^2}\right). \tag{16}$$

At $s = 1/2$, $A = Z = 1$ it agrees with the corresponding results [15, 16] for muonium (where the effective cut-off Λ is provided by the muon mass) and hydrogen (where the integral is cut off at the typical hadronic scale m_ρ).

In the case of deuterium ($s = 1$, $g = \mu_d = 0.857$, $A = 2$, $Z = 1$), we are mainly interested in, the integration over the momentum transfer k is cut off at the inverse deuteron size $\kappa = 45.7$ MeV. In this way we obtain the following result for the relative correction in deuterium:

$$\Delta_{el} = \frac{3\alpha}{8\pi}\frac{m_e}{m_p} \log\frac{\kappa}{m_e}\left(\mu_d - 2 - \frac{3}{\mu_d}\right). \tag{17}$$

At larger momentum transfers, $k > \kappa$, the amplitude of the Compton scattering on a deuteron is just the coherent sum of those amplitudes on free proton and neutron. This contribution to the hf structure can be also easily obtained with the above formulae. Since both nucleons are in the triplet state, one can substitute $\vec{s}/2$ both for $\vec{s}_p$ and for $\vec{s}_n$. With the logarithmic integral cut off here at the usual hadronic scale $m_\rho = 770$ MeV, we get in this way

$$\Delta_{in} = \frac{3\alpha}{4\pi}\frac{m_e}{m_p}\log\frac{m_\rho}{\kappa}\frac{1}{\mu_d}(\mu_p^2 - 2\mu_p - 3 + \mu_n^2). \qquad (18)$$

Here $\mu_p = 2.79$ and $\mu_n = -1.91$ are the proton and neutron magnetic moments.

In the conclusion of this section let us mention strong numerical cancellation between Δ_{el} and Δ_{in}.

4 Contribution of deuteron virtual excitations

The low-energy Compton amplitude discussed above is just linear term of the full amplitude expansion in ω (and in $|\vec{k}|^2/\omega$ for virtual photons). One may expect, however, that for the deuteron with its small binding energy this approximation is insufficient even in the atomic problem considered here. And indeed, we will see below that the deuteron virtual excitations are far from being inessential to our problem, they strongly dominate the effect discussed. Since the contribution of large momentum transfers $k > \kappa$ has been calculated already (see formula (18), we confine now to the region $k < \kappa$. All calculations below are performed in the zero-range approximation (zra) for the deuteron which allows to obtain all the results in a closed analytical form.

Let us start with the transitions induced by the spin interaction only. The corresponding scattering amplitude is

$$M_{4mn} = -\left(\frac{e}{2m_p}\right)^2 \sum_n \left\{ \frac{<0|[\vec{k}\times\vec{S}]_m|n><n|[\vec{k}\times\vec{S}^\dagger]_n|0>}{\omega - E_n - I} - \frac{<0|[\vec{k}\times\vec{S}^\dagger]_n|n><n|[\vec{k}\times\vec{S}]_m|0>}{\omega + E_n + I} \right\} \qquad (19)$$

Here $I = \kappa^2/m_p$ is the deuteron binding energy, $E_n = p^2/m_p$ is the energy of the intermediate state $|n>$ (all intermediate states belong to the continuous spectrum), and

$$\vec{S} = \mu_p\vec{\sigma}_p e^{i\vec{k}\vec{r}/2} + \mu_n\vec{\sigma}_n e^{-i\vec{k}\vec{r}/2}$$

where $\vec{\sigma}_{p(n)}$ is the proton (neutron) spin operator.

When calculating this contribution, we will retain only terms logarithmic in the parameter $\epsilon = I/\kappa = \kappa/m_p \ll 1$. The log is generated by the integration over k, and to obtain it it is sufficient to put the exponents in $\vec{S}$ equal to unity. Then the operator $\vec{S}$ can induce only M1 transitions. In the zra the deuteron ground state is pure ${}^3S_1$ from which M1 transition is possible also to S-states only. But due to the orthogonality of the radial wave functions of different triplet states, the intermediate states confine to ${}^1S_0$.

Since the total spin operator $\vec{s} = (1/2)(\vec{\sigma}_p + \vec{\sigma}_n)$ does not induce triplet-singlet transitions, the operator $\vec{S}$ reduces here to

$$\vec{S} \to (\mu_p - \mu_n)\frac{1}{2}(\vec{\sigma}_p - \vec{\sigma}_n).$$

In our problem of hf structure we need the antisymmetric part of tensor (19) which is linear in the deuteron spin $\vec{s}$. It looks now as follows:

$$M^1_{mn} = -\left(\frac{e}{2m_p}\right)^2 (\mu_p - \mu_n)^2\, i\,\epsilon_{mnk}k_k(\vec{k}\cdot\vec{s})\,\omega \times \int \frac{d\vec{p}}{(2\pi)^3}\frac{|<{}^1S_0, p|{}^3S_1>|^2}{\omega^2 - (p^2+\kappa^2)^2/m_p^2} \qquad (20)$$

where $<{}^1S_0, p|{}^3S_1>$ is the overlap integral of the ground state zra wave function

$$\psi_0 = \sqrt{\frac{\kappa}{2\pi}}\frac{e^{-\kappa r}}{r} \qquad (21)$$

with the singlet one of the momentum p.

This contribution to the electron-deuteron scattering amplitude

$$T^1_{in} = 4\pi\alpha i \int \frac{d^4k}{(2\pi)^4}\frac{d_{im}d_{jn}}{k^4}\frac{\gamma_i(\hat{l} - \hat{k} + m_e)\gamma_j}{k^2 - 2lk} M^1_{mn} \qquad (22)$$

is easily calculated with the logarithmic accuracy. Indeed, to this accuracy the energy denominator in formula (20) can be simplified to

$$\frac{1}{\omega^2 - \kappa^4/m_p^2}.$$

Then the integration over $\vec{p}$ reduces to the completeness relation. The resulting relative correction to the deuterium hf structure is

$$\Delta^1_{in} = \frac{3\alpha}{8\pi}\frac{m_e}{m_p}\log\frac{m_p}{\kappa}\frac{(\mu_p-\mu_n)^2}{\mu_d}. \tag{23}$$

Let us consider at last the inelastic contribution induced by the combined action of the convection and spin currents. Since the convection current is spin-independent, all the intermediate states are triplet ones as well as the ground state. Therefore, here operator $\vec{S}$ simplifies to

$$\vec{S} \to \vec{s}(\mu_p e^{i\vec{k}\vec{r}/2} + \mu_n e^{-i\vec{k}\vec{r}/2}). \tag{24}$$

According to the common selection rules, neither $^3S_1$ can be excited by the convection current from the ground state. But in the zra all the states with $l \neq 0$ are free ones. It means that here we can use as the intermediate states just plane waves, eigenstates of the momentum operator. Then the only matrix element entering the amplitude is

$$<0|e^{\pm i\vec{k}\vec{r}/2}|\vec{p}> = \frac{\sqrt{8\pi\kappa}}{(\vec{p}\pm\vec{k}/2)^2+\kappa^2}. \tag{25}$$

In this way the amplitude itself simplifies to

$$M^2_{mn} = \left(\frac{e}{2m_p}\right)^2 2\kappa\omega\int\frac{d\vec{p}}{\pi^2}\left\{\frac{\mu_p}{[(\vec{p}-\vec{k}/2)^2+\kappa^2]^2}\right.$$

$$\left.+\frac{\mu_n}{[(\vec{p}-\vec{k}/2)^2+\kappa^2][(\vec{p}+\vec{k}/2)^2+\kappa^2]}\right\}$$

$$\times\frac{(2p-k/2)_m\, i\epsilon_{nrs}k_r s_s - (2p-k/2)_n\, i\epsilon_{mrs}k_r s_s}{\omega^2-(p^2+\kappa^2)^2/m_p^2}. \tag{26}$$

Integrals we come across when calculating the corresponding part of the electron-deuteron scattering amplitude

$$T^2_{in} = 4\pi\alpha i\int\frac{d^4k}{(2\pi)^4}\frac{d_{im}d_{jn}}{k^4}\frac{\gamma_i(\hat{l}-\hat{k}+m_e)\gamma_j}{k^2-2lk}M^2_{mn}, \tag{27}$$

are rather tedious even when we confine to terms singular in the parameter $\epsilon = \kappa/m_p \ll 1$: $1/\epsilon$ and $\log\epsilon$. This relative correction to the hf structure is to this approximation

$$\Delta^2_{in} = \alpha\frac{m_e}{2\kappa}\frac{\mu_p-\mu_n}{\mu_d} - \frac{3\alpha}{\pi}\frac{m_e}{m_p}\log\frac{m_p}{\kappa}\frac{\mu_p-\mu_n}{\mu_d}. \tag{28}$$

One of the terms in this correction,

$$-\alpha\frac{m_e}{2\kappa}\frac{\mu_n}{\mu_d},$$

was obtained and discussed in Refs. [4, 5, 6, 7]. However, another term

$$\alpha\frac{m_e}{2\kappa}\frac{\mu_p}{\mu_d},$$

is larger numerically.

5 Corrections due to finite distribution of the deuteron charge and magnetic moment

In the case of hydrogen this problem was considered many years ago [17]. For deuterium those corrections should be obviously larger. In the zra the problem has here a closed solution.

Let us start with the second-order amplitude of the electron-deuteron scattering as induced by the deuteron charge and magnetic moment. The nucleus will be treated in the static limit. However, its finite charge and magnetic moment distributions will be taken into account by introducing the corresponding form-factors, F_{ch} and F_m. This amplitude is

$$V = -(4\pi\alpha)^2\frac{\mu_d}{2m_p}\int\frac{d\vec{q}}{(2\pi)^3}\, i[\vec{s}\times\vec{q}]\,\frac{F_{ch}(\vec{q}^2)F_m(\vec{q}^2)}{\vec{q}^4}$$

$$\times\frac{\gamma_0(\hat{l}+\hat{q}+m_e)\vec{\gamma}-\vec{\gamma}(\hat{l}+\hat{q}+m_e)\gamma_0}{(l+q)^2-m_e^2}. \tag{29}$$

Here again $l_\mu = (m_e, 0, 0, 0)$, and $q_\mu = (0, \vec{q})$. This expression can be conveniently transformed to

$$V = \frac{8m_e\alpha}{\pi}\int\frac{dq}{q^2}F_{ch}(q^2)F_m(q^2)T_0 \tag{30}$$

where T_0 is the momentum-independent magnetic Born amplitude (15).

The effect we are interested in, vanishes of course if unity is substituted for both form-factors. Therefore, the corresponding relative correction to the Born amplitude T_0 and to the hf splitting is in fact

$$\Delta_f = \frac{8m_e\alpha}{\pi}\int\frac{dq}{q^2}\left[F_{ch}(q^2)F_m(q^2)-1\right]. \tag{31}$$

In the zra both deuteron form-factors have simple form:

$$F_{ch}(q^2) = F_m(q^2) = <0|e^{i\vec{q}\vec{r}/2}|0> = \frac{4\kappa}{q}\,\mathrm{arctg}\frac{q}{4\kappa}. \qquad (32)$$

Substituting it into formula (31), we get the following explicit expression for the correction discussed:

$$\Delta_f = -\alpha\,\frac{m_e}{3\kappa}(1+2\log 2). \qquad (33)$$

Two closely related features of the effect (in no way specific for deuterium only) are worth emphasizing here. This correction is of first (but not second) order in the ratio of the nuclear size to the Bohr radius $m_e\alpha/\kappa$. Then, contrary to possible naive expectations, the contributions of the charge and magnetic form-factors are not additive. Both circumstances can be traced back to the fact that the typical momenta entering integral (31) are of the nuclear, but not atomic, scale.

6 Deuterium hf structure, discussion of results

Our total result for the nuclear-structure corrections to the deuterium hf structure, comprising all the contributions, (17), (18), (23), (28), (33), is

$$\Delta = \alpha\,\frac{m_e}{2\kappa}\left\{\frac{\mu_p-\mu_n}{\mu_d} - \frac{2}{3}(1+2\log 2)\right\} + \frac{3\alpha}{8\pi}\frac{m_e}{m_p}\log\frac{m_p}{\kappa}\,\frac{(\mu_p-\mu_n)^2}{\mu_d}$$
$$-\frac{3\alpha}{\pi}\frac{m_e}{m_p}\log\frac{m_p}{\kappa}\,\frac{\mu_p-\mu_n}{\mu_d} + \frac{3\alpha}{8\pi}\frac{m_e}{m_p}\log\frac{\kappa}{m_e}\,\frac{1}{\mu_d}(\mu_d^2-2\mu_d-3) \qquad (34)$$
$$+\frac{3\alpha}{4\pi}\frac{m_e}{m_p}\log\frac{m_p}{\kappa}\,\frac{1}{\mu_d}(\mu_p^2-2\mu_p-3+\mu_n^2).$$

Numerically this correction to the hf splitting in deuterium constitutes

$$\Delta\nu = 43\,\mathrm{kHz}. \qquad (35)$$

It should be compared with the lacking 45 kHz (see (3)). Taking into account the approximations made, first of all the crude nuclear model (zra), then the neglect of nonlogarithmic contributions, we believe that the agreement is quite satisfactory. In particular, including the correction due to the effective interaction radius r_0 into the normalization of the deuteron ground

state wave function (see details in the next section) would certainly enhance some contributions.

Clearly, the nuclear effects discussed are responsible for the bulk of the difference between the pure QED calculations and the experimental value of the deuterium hf splitting. The calculation of this hf correction, including accurate treatment of nuclear effects, would serve as one more sensitive check of detailed models of deuteron structure.

7 Nuclear polarizability and Lamb shift in deuterium

The nuclear polarizability contribution to the Lamb shift was considered recently in Refs. [8, 9, 10]. Here we present an analytical calculation of the effect with a closed result. The zra approximation used by us is applicable when the region of the wave function localization is much larger than the interaction range. Essentially the same condition is necessary to use, instead of the true interaction, the crude approximation of the square-well potential, as it is done in Refs. [8, 9].

The effect we are interested in now, is due to the photon-deuteron scattering amplitude induced by the convection current only. We will see that in this problem, that of the nuclear polarizability contribution to the Lamb shift, the characteristic values of the photon 4-momenta are as follows:

$$\omega,\ |\vec{k}| \le I = \frac{\kappa^2}{m_p} \ll \kappa \sim |\vec{p}|. \qquad (36)$$

Therefore, now we can omit in the Compton amplitude all dependence on $\vec{k}$. As well as in the amplitude M^2_{mn}, all the intermediate states here have $l \neq 0$ and can be described therefore by plane waves. We will use again matrix element (25), but this time at $\vec{k}=0$. At last, in the present problem we are interested in the scalar part of scattering amplitude which reduces to

$$-\left(\frac{e}{2m_p}\right)^2\frac{4}{3}\,\delta_{mn}\,\kappa\int\frac{d\vec{p}}{\pi^2}\frac{p^2}{(p^2+\kappa^2)^2}$$
$$\times\left\{\frac{1}{\omega-(p^2+\kappa^2)/m_p} - \frac{1}{\omega+(p^2+\kappa^2)/m_p}\right\}.$$

We subtract from the expression in braces the term

$$-2\,\frac{m_p}{p^2+\kappa^2}.$$

After integrating over $\vec{p}$ this term being added to the Thomson scattering (seagull) amplitude for a proton, $-e^2/m_p$, reproduces the correct one for a deuteron, $-e^2/2m_p$. With the identity

$$\frac{1}{\omega-u}-\frac{1}{\omega+u}+\frac{2}{u}=\frac{2\omega^2}{(\omega^2-u^2)\,u}$$

we get the following expression for the photon-deuteron scattering amplitude in question:

$$M^3_{mn} = -\left(\frac{e}{2m_p}\right)^2 \frac{8}{3}\,\delta_{mn}\,\kappa\,\omega^2 m_p$$

$$\times \int \frac{d\vec{p}}{\pi^2}\,\frac{p^2}{(p^2+\kappa^2)^3\,\{\omega^2-(p^2+\kappa^2)^2/m_p^2\}} \qquad (37)$$

The contribution of this tensor to the electron-deuteron scattering amplitude

$$T^3_{in} = 4\pi\alpha i \int \frac{d^4k}{(2\pi)^4}\,\frac{d_{im}d_{jn}}{k^4}\,\frac{\gamma_i(\hat{l}-\hat{k}+m_e)\gamma_j}{k^2-2lk}\,M^3_{mn} \qquad (38)$$

can be easily transformed to

$$T^3_{in} = \frac{32\pi^2\alpha^2\kappa}{3m_p}\int \frac{d\vec{p}}{\pi^2}\,\frac{p^2}{(p^2+\kappa^2)^3}$$

$$\times\, i\int \frac{d^4k}{(2\pi)^4}\left\{\frac{1}{\omega\,(k^2-2lk)}+\frac{2\,\omega^3}{k^4\,(k^2-2lk)}\right\}\frac{1}{\omega^2-(p^2+\kappa^2)^2/m_p^2}. \qquad (39)$$

The first term in the braces here contains no photon propagation, neither $1/k^4$, nor $1/k^2$. In other words, it corresponds to the instantaneous Coulomb interaction. The second term corresponds to the exchange by three-dimensionally transverse quanta, i.e., to the magnetic interaction of convection currents.

Perhaps, the most convenient succession of integrating expression (39) is as follows: the Wick rotation; transforming the integral over the Euclidean ω to the interval $(0,\infty)$; the substitution $\vec{k}\to\vec{k}\,\omega$; integration over ω; integration over $\vec{k}$ (at the last two procedures it gets clear that the effective values of ω, $|\vec{k}|$ belong to interval (36) indeed); at last, integration over p. The following identity is useful here:

$$\int_0^1 dx\,(1-x)^{a-1}x^{b-1}\log x = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}\,[\psi(b)-\psi(a+b)];\quad \psi(b)=\frac{d}{db}\log\Gamma(b).$$

The effective electron-nucleus interaction operator (equal to $-T^3_{in}$) can be finally presented in the coordinate representation as

$$V_{le} = -\alpha\, m_e\, \alpha_d(0)\, 5\left(\log\frac{8I}{m_e}+\frac{1}{20}\right)\delta(\vec{r}). \qquad (40)$$

Here $\alpha_d(0)$ is the static value of the deuteron electric polarizability defined as usual by the relation

$$\alpha_d(\omega) = 4\pi\alpha\,\frac{2}{3}\int\frac{d\vec{p}}{(2\pi)^3}\,\frac{p^2+\kappa^2}{m_p}\,\frac{<0|\vec{r}|n><n|\vec{r}|0>}{(p^2+\kappa^2)^2/m_p^2-\omega^2}. \qquad (41)$$

The matrix elements entering expression (41) are dominated by large distances. In this asymptotic region the naive zra expression (21) for the deuteron ground state wave function should be supplied by the correction factor $(1-r_0\kappa)^{-1/2}$ taking into account finite effective interaction radius r_0 (see Refs. [14, 18]). In this way we get the following result for the static electric polarizability:

$$\alpha_d(0) = \frac{\alpha}{32\,(1-r_0\kappa)}\,\frac{m_p}{\kappa^4} = 0.64\,\text{fm}^3. \qquad (42)$$

This numerical value is close to the experimental one [19]: $0.70(5)\,\text{fm}^3$ (as well as to the values 0.613, 0.623, $0.625\,\text{fm}^3$ obtained in Ref. [10] with different separable nuclear potentials and to $0.635\,\text{fm}^3$ found in Ref. [8] with a square-well potential).

The overall result (40) consists of two contributions of different physical origin. The dominating one is generated by the instantaneous Coulomb interaction. Its contribution to the overall numerical factor

$$-5\left(\log\frac{8I}{m_e}+\frac{1}{20}\right)$$

in formula (40) is

$$-4\left(\log\frac{8I}{m_e}+\frac{5}{12}\right).$$

The magnetic interaction contributes to the overall factor

$$-\left(\log\frac{8I}{m_e}-\frac{17}{12}\right).$$

The level shift of the deuterium ground state produced by operator (40) constitutes -22.3 kHz. The Coulomb and magnetic contributions to it are, respectively, -19.7 and -2.6 kHz. The results are close to those of Refs. [8, 9, 10].

No wonder that the Coulomb contribution is negative: this is a true second-order (in the electron-nucleus static interaction) correction to the ground state of a system consisting of an electron at rest and a nucleus which is in the ground state itself. The sign of the magnetic contribution cannot be fixed in this way: in the language of the common noncovariant perturbation theory this is a fourth-order correction, second-order in the photon-electron interaction and second-order in the photon-nucleus one.

One more contribution to the Lamb shift in deuterium is caused by the deuteron magnetic polarizability, considered earlier also in Ref. [10]. This is in fact the contribution of the scalar part of amplitude (19). The calculation simplifies due to the following circumstances. First, the numerators d_{im} of the photon propagators reduce here obviously to δ_{im}. Then, the integration over $\vec{k}$ is spherically-symmetric one. So, to our purpose the scalar part of amplitude (19) may be simplified to

$$M^4_{mn} = -4\pi\alpha\,\frac{(\mu_p-\mu_n)^2\,\kappa\,(\kappa+\kappa_1)^2}{9\,m_p^3}$$

$$\times\,\delta_{mn}\,\vec{k}^2\int\frac{d\vec{p}}{\pi^2}\,\frac{1}{(p^2+\kappa^2)(p^2+\kappa_1^2)\,[\omega^2-(p^2+\kappa^2)^2/m_p^2]}. \tag{43}$$

We have used here the explicit form of the $^1S_0$ coordinate wave function in the zra:

$$\psi_s = \frac{\sin(pr+\delta)}{\sqrt{2}\,\pi r} \tag{44}$$

where

$$\mathrm{ctg}\,\delta = \frac{\kappa_1}{p},\quad \kappa_1 = 7.9 MeV.$$

Its overlap with the ground state zra wave function (21) constitutes

$$<{}^1S_0|^3S_1> = \frac{\sqrt{8\pi\kappa}\,(\kappa+\kappa_1)}{(p^2+\kappa^2)\sqrt{p^2+\kappa_1^2}}. \tag{45}$$

Further calculations are close to those related to the electric polarizability; only the last integration, that over p, is done numerically for the nonlogarithmic contribution. The resulting effective electron-nucleus interaction operator can be written as

$$V_{lm} = \alpha\,m_e\,\beta_d(0)\left(\log\frac{8I}{m_e}-1.24\right)\delta(\vec{r}). \tag{46}$$

Here $\beta_d(0)$ is the static value of the deuteron magnetic polarizability equal to

$$\beta_d(0) = \frac{\alpha\,(\mu_p-\mu_n)^2}{8\,m_p\,\kappa^2}\,\frac{1+\kappa_1/3\kappa}{1+\kappa_1/\kappa}. \tag{47}$$

This contribution to the Lamb shift of the deuterium ground state constitutes 0.31 kHz which is very close to the result of Ref. [10].

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