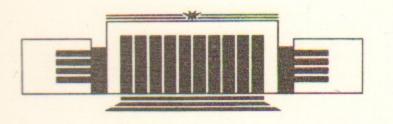S.V. Faleev, P.G. Silvestrov

INSTANTON–ANTI-INSTANTON PAIR INDUCED CONTRIBUTION
TO $R_{e+e-\to HADRONS}$ AND $R_{\tau\to HADRONS}$

НОВОСИБИРСК

Instanton–anti-instanton pair induced contributions to $R_{e^+e^-\to hadrons}$ and $R_{\tau\to hadrons}$

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Abstract

The instanton–anti-instanton pair induced asymptotics of perturbation theory expansion for the cross section of electron–positron pair annihilation to hadrons and hadronic width of τ-lepton was found. For $N_f = N_c$ the nonperturbative instanton contribution is finite and may be calculated without phenomenological input. The instanton induced peturbative asymptotics was shown to be enhanced as $(n + 10)!$ and in the intermediate region $n < 15$ may exceed the renormalon contribution. Unfortunately, the analysis of $\sim 1/n$ corrections shows that for $n \sim 10$ the obtained asymptotic expressions are at best only the order of magnitude estimate. The asymptotic series for $R_{e^+e^-\to hadrons}$, though obtained formally for $N_f = N_c$, is valid up to energies $\sim 15\text{Gev}$. The instanton–anti-instanton pair nonperturbative contribution to $R_{\tau\to hadrons}$ blows up. On the one hand, this means that instantons could not be considered *ab–initio* at such energies. On the other hand, this result casts a strong doubt upon the possibility to determine the α_s from the τ–lepton width.

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1 Introduction. Instanton – Renormalon

The semiclassical technique for high order perturbation theory estimates was suggested by L.N.Lipatov [1] almost 20 years ago. The main advantage of this approach is that there is no need to deal with the Feynman graphs in order to find the asymptotics. Only the contribution of one "the most important" classical configuration in the functional space (instanton) is considered, which can be associated with the sum of a huge number of these graphs.

Shortly after the Lipatov's paper was published another mechanism for the factorial growth of the series of perturbation theory was found [2, 3]. It turns out that in the theories with the running coupling constant a very simple chain of graphs may dominate in the asymptotics of perturbation theory. By analogy with the instanton this chain of graphs was called renormalon.

In the present work we will calculate the instanton induced contribution to the asymptotics of perturbation theory series for the cross section of electron-positron pair annihilation and hadronic width of τ–lepton. As we shell see, at $n < 15$ the instanton contribution to the n-th order may be quite competitive with the renormalon contribution.

During the last few years the issue of the asymptotic behaviour of perturbation theory in QCD and QED (see e.g. [4]-[9]) has attached the renewed interest. However, the main attention was paid to the renormalon asymptotics.

There are two types of renormalons – ultraviolet and infrared [2]. The usual form of the ultraviolet renormalon contribution to, for example,

$R_{e+e-\rightarrow hadrons}$ is:

$$R_{e+e-\rightarrow hadrons} = \sum R_n \left(\frac{\alpha_s}{4\pi}\right)^n \ , \ R_n \sim (-b)^n \, n! \ , \tag{1}$$

where $b = \frac{11}{3} N_c - \frac{2}{3} N_f \approx 10$. In QCD the series (1) is sign alternating and at least the Borel sum of the series is well defined. The problems with summation of ultraviolet renormalon appear in QED, where the coefficient b is negative.

Generally speaking, the infrared renormalon contribution to the asymptotics depends on the process considered. For $R_{e+e-\rightarrow hadrons}$ and $R_{\tau\rightarrow hadrons}$ the contribution of the first infrared renormalon to the asymptotics reads:

$$R_n \sim \left(\frac{b}{2}\right)^n n! \ . \tag{2}$$

All problems associated with the asymptotic series become more clear if the physical quantities are presented in the form of Borel integral:

$$R_{e+e-} = \int_0^\infty \exp\left(-\frac{4\pi}{\alpha} t\right) F(t) dt \ . \tag{3}$$

Now the singularities of the function $F(t)$ on the complex plane are the natural sources of asymptotic series. The first ultraviolet renormalon in QCD (1) corresponds to the singularity at negative value $t = -1/b$. The first singularity associated with infrared renormalon (2) is located twice father from the origin, but at positive value $t = 2/b$. Thus, the terms of the series (2) grow with n more slowly than those of the series (1). On the other hand, for the infrared renormalon one has to choose the correct way of the integration over singularity which lies on the positive real axis. Different definitions of the integral (3) will lead to the different power corrections $\delta R \sim 1/q^4$.

The issue of infrared renormalon is directly related with the well known problem of Landau pole in theories with running coupling. The singularity of Borel transform $F(t)$ (3) associated with infrared renormalon coincides with Landau pole which arises when the exchange of soft gluon in fermion loop is taken into account (see the example of renormalon type graphs in any of the papers [2]-[10]). At this point one may say that the main unsolved problems of QCD (confinement) are also hidden in the infrared renormalon [3].

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[3] In fact, the situation with ultraviolet renormalon is also not very clear. Being written more accurately than (1), the ultraviolet renormalon contribution takes the form:

$$R_n = An^\gamma \, (-b)^n \, n! \ , \tag{4}$$

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As we have already said, another approach for high order perturbation theory estimates based on the consideration of the specific ("classical") large fluctuations in the functional space was suggested by Lipatov. The natural example of such important fluctuations in QCD is the instanton–anti-instanton pair. The generic form of instanton induced asymptotics appears to be:

$$R_{nIA} \sim (n + 4N_c)! \tag{5}$$

(here we do not show either overall numerical factor, which also may be sufficiently large, or the factor n^γ with $\gamma \sim 1$). On the Borel plane the instanton - anti-instanton pair corresponds to the pole at rather distant point $t = 1$, but residue in the pole turns out to be very large $\sim (4\pi/\alpha_s)^{4N_c}$.

Thus, we can see that though the renormalons (1,2) do dominate at very large n ($n > 15$), the instanton-induced contribution may dominate in the intermediate asymptotics $n = 5 \div 15$. If so, the pure renormalon behavior (1) will never be observed in directly calculated terms of perturbation theory due to a strong competition with the instanton contribution.

It is clear, that the exactly known $3 \div 4$ terms of perturbation theory series (for β-function, $R_{e+e-\rightarrow hadrons}$ or $R_{\tau\rightarrow hadrons}$) are much smaller than the estimate (5) and the question at what number n the perturbative series could reach the full strength (5) is open now. The only way to answer this question is to consider the $\sim 1/n$ corrections to the leading asymptotics. In this paper we have found (and summed up) two subseries of corrections to R_{nIA}. The first one behaves like $(N_c^2 \ln (n)/n)^k$, the second - like $(N_c^2 \ln (N_c)/n)^k$. The final expressions for R_{nIA} are given below (46), (53). The $\sim 1/n$ corrections tend to decrease the asymptotic predictions and thus improve the agreement with the "experiment". But important is not this agreement. Unfortunately, the corrections look like $\sim N_c^2/n$ and, thus, even for $n \sim 10$ our asymptotic expression can be used at best as the estimate of the order of magnitude.

Up to now the instanton induced asymptotics of perturbation theory for $R_{e+e-\rightarrow hadrons}$ was considered only in the paper of I.I.Balitsky [11]. Many of the technical methods used by Balitsky were useful in our work, but nevertheless, he could not find correctly either asymptotics of perturbation theory, or nonperturbative contribution of the instanton - anti-instanton pair in the most actual case $N_f = N_c$. The instanton induced contribution to the

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where the constant $\gamma \sim 1$. It has been believed for a long time that the main contribution to the ultraviolet renormalon comes from the graphs with exchanging of one dressed gluonic line as in the case of infrared renormalon. However, in the recent work [10] Vainshtain and Zakharov showed, that the contribution of these graphs is small compared to the graphs with two, three etc. dressed lines. As a result, only the constant γ is known now and it is absolutely unclear how to search for the overall factor A.

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perturbative asymptotics for the simple correlator of two gluonic currents $j \sim \alpha_s [G_{\mu\nu}^a]^2$ has also been considered by one of the authors of the present paper in recent work [16].

Similarly to the infrared renormalon (2) all terms of the series (5) are of the same sign. Nevertheless, the problem of summing the series (5) seems not as hopeless as that of the renormalon. Following G.'t Hooft [2], the author of [11] proposed to rewrite the integral over the instanton–anti-instanton pair in the Borel form by considering the action as a collective variable. Within this approach the well–separated instanton–anti-instanton pair is responsible for the singular part of Borel function, while the ambiguous strongly interacting instanton and anti-instanton contribute to its smooth part. On the other hand, the best way to describe the smooth part of the Borel transform is to calculate exactly the few first terms of perturbative expansion. The accurate subtraction from the singularity of dilute gas contribution in the toy model (double well oscillator) allowed us to find the finite nonperturbative instanton–anti-instanton contribution [15]. In QCD at $N_f = N_c$ the Borel integral diverges only logarithmically and the total nonperturbative contribution from instanton–anti-instanton pair may be found by cutting the instanton size at $\rho \ll 1/\Lambda_{QCD}$.

In the next section we combine some useful results concerning the instanton–anti-instanton pair and the behavior of light fermions in instanton background. The correlation function of two electro-magnetic currents is calculated in section 3. At $N_f = N_c$ we calculate the finite nonperturbative instanton contribution as well as the instanton induced asymptotics of the perturbation theory. The issue of $\sim 1/n$ corrections to the asymptotics is also considered. Unfortunately, as we have already mentioned, the asymptotic expression for the instanton contribution may be considered at best only as the estimate of the order of magnitude up to $n \sim 10$. On the other hand if n is not very large the instanton may make completely unvisible the renormalon contribution. The possibility, discussed in section 4, to extrapolate our asymptotic formulas for $R_{e+e-\to hadrons}$, found for $N_f = N_c$, up to energy $\sim 10 - 15 Gev$ may be considered as a poor consolation. At least at such energies the high orders of the perturbation theory are not lost at the background of standard power corrections. More important is the issue of instanton contribution to $R_{\tau \to hadrons}$ (section 5) due to numerous attempts to extract from this quantity the value of $\alpha_s(m_\tau)$. Unfortunately, here our result is negative too. Even if one does not take into account the asymptotics of the perturbation theory (which also has to be reached), the pure nonperturbative contribution to $R_{\tau \to hadrons}$ turns out to be one or two orders of magnitude larger than one can admit. In spite of the fact, that at energy

$m_\tau c^2 \sim 2 Gev$ our result is again only the estimate of the order of magnitude, we have a strong doubt about the possibility to find the accurate value of α_s from the τ–lepton width.

2   The Instanton–anti-instanton pair

The interesting physical quantities, like, e.g., $R_{e+e-\to hadrons}$, may be naturally found via the analytical continuation of the corresponding Euclidean correlators:

$$R_{e+e-\to hadrons}(q^2) = 12\pi Im\, \Pi(-q^2 - i\varepsilon) , \qquad (6)$$

$$\Pi_{\mu\nu}(q^2) = \Pi(q^2)(q_\mu q_\nu - q^2 \delta_{\mu\nu}) = \int d^4 x e^{iqx} \langle j_\mu(x) j_\nu(0) \rangle ,$$

where $j_\mu = \sum_{flavors} e_f \Psi_f^+ \gamma_\mu \Psi_f$. Therefore, we have to calculate the contribution of the instanton–anti-instanton pair to the correlator. As we have said above, the strongly interacting instanton and anti-instanton contribute only to the smooth part of the Borel function F (3) and do not effect the asymptotics of perturbation theory. The singular part of the Borel function is saturated by the almost non-interacting pseudoparticles. That is why, one can obtain the reliable prediction for the asymptotics starting from such ill defined object as the instanton–anti-instanton pair. More concretely, the field configuration relevant for the large orders of perturbation theory is a small instanton inside of a very large anti-instanton (or vice versa). The size of small instanton is regulated by the internal momentum in correlator (6) $q\rho_I \sim 1$. The size of anti-instanton (as well as the distance between the centers of pseudoparticles $R \sim \rho_A \gg \rho_I$) determines, how close we are to the singularity on the Borel plane.

We are interested in the instanton–anti-instanton interaction in the leading approximation. Therefore, the simple sum of instanton and anti-instanton may be used:

$$A_\mu = U_A A_\mu^A U_A^+ + U_I A_\mu^I U_I^+ , \qquad (7)$$

where U_A, U_I are the constant $SU(N_c)$ matrices in the color space. By trivial gauge rotation one can make $U_I = 1$. For "small" instanton the singular gauge seems to be preferable:

$$A_\mu^I = \frac{\overline{\eta}_{\mu\nu}(x - x_I)_\nu \rho_I^2}{(x - x_I)^2((x - x_I)^2 + \rho_I^2)} , \qquad (8)$$

where $\overline{\eta}_{\mu\nu} \equiv \tau^a \overline{\eta}_{\mu\nu}^a$, and τ^a are the usual Pauly matrices located in the upper left 2×2 corner (another elements vanish) of $N \times N$ matrix describing the gluonic fields.

Before we add the anti-instanton to (8), the singularity at $x = x_I$ is pure gauge singularity. In order to suppress the unphysical singularities in the sum (7), one may choose A_μ^A in any regular gauge which leads to $A_\mu^A(x = x_I) = 0$. For example, one may slightly rotate the BPST anti-instanton:

$$A_\mu^A = S\left[\frac{\overline{\eta}_{\mu\nu}(x - x_A)_\nu}{(x - x_A)^2 + \rho_A^2}\right]S^+ + iS\partial_\mu S^+ . \qquad (9)$$

Near the center of instanton the matrix $S(x)$ satisfying the above condition has the form:

$$S = e^{i\Theta} , \quad \Theta = B_\mu(x - x_I)_\mu + C_{\mu\nu}(x - x_I)_\mu(x - x_I)_\nu , \qquad (10)$$

where $B_\mu = -\frac{\overline{\eta}_{\mu\nu}R_\nu}{R^2 + \rho_A^2} = -A_\mu^A(x = x_I)$, $R_\mu = (x_A - x_I)_\mu$ and $C_{\mu\nu} = C_{\nu\mu}$ is an arbitrary symmetric tensor.

After direct calculation the classical action of the instanton–anti-instanton configuration may be found with the usual dipole–dipole interaction of pseudoparticles:

$$S_{IA} = \frac{4\pi}{\alpha_s}\{1 - \xi h\} , \quad \xi = \frac{\rho_I^2 \rho_A^2}{(R^2 + \rho_A^2)^2} , \quad h = 2|TrO|^2 - TrOO^+ , \qquad (11)$$

and O is the upper left 2×2 corner of the matrix $U = U_A^+ U_I$ (7).

The features of light fermions are mostly sensitive to the presence of instantons. The Dirac operator $\hat{D}$ for each flavor of massless fermions has two eigenfunctions $\Psi_\pm$ with anomalously small eigenvalues $\lambda_\pm$. As we consider the case of almost noninteracting pseudoparticles it is natural to search for these eigenfunction in the form of linear combination of nonperturbed zero modes of separate pseudoparticles. Explicit expressions for zero modes in the background of singular instanton and regular anti-instanton are:

$$\Psi_I = \frac{1}{\pi}\frac{\rho_I}{[x^2 + \rho_I^2]^{3/2}}\frac{x_\mu \tau_\mu^-}{|x|}\begin{pmatrix}\phi \\ \phi\end{pmatrix} , \quad \Psi_A = \frac{1}{\pi}\frac{\rho_A}{[x^2 + \rho_A^2]^{3/2}}U\begin{pmatrix}\phi \\ -\phi\end{pmatrix} , \quad (12)$$

where $\phi^{\alpha m} = \varepsilon_{\alpha m}/\sqrt{2}$ for $\alpha = 1, 2$ and $\phi^{\alpha m} = 0$ for $\alpha > 2$, α is color index, $m = 1, 2$ is spinor index, $\varepsilon_{\alpha m}$ is an antisymmetric tensor, and $\tau^\pm = (\mp i, \vec{\tau})$. All correlation functions which are of interest for us are saturated by the

region $|x - x_I| \sim \rho_I$. Within this region the anti-instanton zero mode should be modified. It is easy to verify that the spinor function

$$\Psi_A = \frac{1}{\pi}\sqrt{\frac{(x - x_I)^2}{(x - x_I)^2 + \rho_I^2}}\frac{\rho_A}{[R^2 + \rho_A^2]^{3/2}}U\begin{pmatrix}\phi \\ -\phi\end{pmatrix} , \qquad (13)$$

is the solution of the Dirac equation $\hat{D}_I\Psi_A = 0$ at $x - x_I \sim \rho_I$ and approaches (12) at $|x - x_I| \gg \rho_I$.

After diagonalization of the Dirac operator $\hat{D}_{I+A}$ within the subspace of zero modes (also the identity $\lambda_+ \equiv -\lambda_-$ may be useful) one gets:

$$\lambda_\pm = \pm\frac{2\rho_I\rho_A}{(\rho_A^2 + R^2)^{3/2}}|TrO| , \quad \Psi_\pm = \frac{1}{\sqrt{2}}\left(\Psi_I \pm \frac{TrO^+}{|TrO|}\Psi_A\right) . \qquad (14)$$

Because the instantons interact very slightly the nonzero modes contribution to the fermionic determinant factorizes. The Green function in this case also has rather simple form:

$$S(x, y) = S_\lambda + G_I + G_A - G_0 + O(\xi) . \qquad (15)$$

Here, $G_0(x - y)$ is the bare Green function, G_I, G_A are Green functions in the background of separate instanton and anti-instanton and S_λ is the zero mode contribution. From (12-14) one gets:

$$S_\lambda(x, y) = \frac{\Psi_+(x)\Psi_+^+(y)}{\lambda_+} + \frac{\Psi_-(x)\Psi_-^+(y)}{\lambda_-} = \qquad (16)$$

$$\frac{1}{|\lambda|}\left\{\frac{TrO^+}{|TrO|}\Psi_A\Psi_I^+ + \Psi_I\Psi_A^+\frac{TrO}{|TrO|}\right\} .$$

The Green function $S(x, y)$ may contain only the terms which convert the right fermions to the left and vice versa. That is why the largest, proportional to $\Psi_I\Psi_I^+$, term in S_λ vanishes. As a result, the contributions of the zero modes and of "quantum" Green function G_I to the correlation functions are of the same order of magnitude. The last two terms in (15) almost cancel $G_A - G_0 \sim \xi$ in the region of interest $|x - x_I| \sim \rho_I \ll \rho_A$. The Green function in the instanton field (for simplicity, we put $x_I = 0$) has been found in [14]:

$$2\pi^2 G_I(x, y) = \frac{i\gamma_\mu}{\sqrt{T_x T_y}}\frac{(x\tau^-)}{|x|}\left[z_\mu\frac{\rho^2 + (\tau^+ x)(\tau^- y)}{z^4} + \right. \qquad (17)$$

$$\left. \tau_\mu^+\frac{(z\tau^-)\rho^2}{2z^2 T_x}\right]\frac{(\tau^+ y)}{|y|}\left(\frac{1 + \gamma_5}{2}\right) + (c.c., x \leftrightarrow y),$$

where $T_x = \rho^2 + x^2$ and $z = x - y$. We use the Hermitean euclidean matrices γ_μ which satisfy $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$ and, for example, the bare Green function satisfies the equation $-i\gamma_\mu\partial_\mu G_0 = \delta(x - y)$.

3 The calculation of the correlation function.

Now, at last, we can write down the expression for the correlation function (6):

$$\Pi_{\mu\nu} = 2\int e^{iqx}\exp\left\{\frac{4\pi}{\alpha_s}\xi h\right\}\left\{-\sum_f e_f^2 T_{\mu\nu}(x,0) + \left(\sum_f e_f\right)^2 B_\mu(x)B_\nu(0)\right\}\times$$

$$[4\xi^{3/2}|TrO|^2]^{N_f}\frac{d(\rho_I)}{\rho_I^5}\frac{d(\rho_A)}{\rho_A^5}dx\,dx_I\,dx_A\,d\rho_I\,d\rho_A\,dU ,\qquad (18)$$

where

$$T_{\mu\nu}(x,y) = Tr\{\gamma_\mu S(x,y)\gamma_\nu S(y,x)\} - Tr\{\gamma_\mu G_0(x,y)\gamma_\nu G_0(y,x)\},(19)$$

$$B_\mu(x) = Tr\{\gamma_\mu S(x,x)\} .$$

The factor 2 in front of the integral in (18) accounts for the equal contribution from small anti-instanton and large instanton. The factor $\sim \xi^{3N_f/2}|TrO|^{2N_f}$ in the square brackets accounts for the contribution of almost zero modes (14) to the fermion determinant. The instanton density reads [18, 19, 20]:

$$d(\rho) = \frac{c_1 e^{-N_c c_2 + N_f c_3}}{(N_c - 1)!(N_c - 2)!}\left(\frac{2\pi}{\alpha_s(\rho)}\right)^{2N_c}\exp\left(-\frac{2\pi}{\alpha_s(\rho)}\right) .\qquad (20)$$

for $\overline{MS}$ scheme $c_1 = 2e^{5/6}/\pi^2$, $c_2 = 1.511$, $c_3 = 0.292$, $c_2 - c_3 = 2\ln 2 - 1/6$. Up to now, in many papers the wrong values of c_2 and c_3 have been used (for which, in particular, $c_2 - c_3 = 2\ln 2$), though the error in [18] which had been done while passing from Pauli–Villars to dimensional regularization was corrected in [20] and t'Hooft in later papers used the correct expression for $d(\rho)$.

We will also use the well known two–loop formula:

$$\frac{4\pi}{\alpha_s(p)} = b\ln\left(\frac{p^2}{\Lambda^2}\right) + \frac{b'}{b}\ln\left(\ln\left(\frac{p^2}{\Lambda^2}\right)\right) + ... ,\qquad (21)$$

where $b = \frac{11}{3}N_c - \frac{2}{3}N_f$ and $b' = \frac{34}{3}N_c^2 - \frac{13}{3}N_f N_c + N_f/N_c$. If $N_f = N_c = 3$ one has $b = 9$ and $b' = 64$.

In order to reduce the integral (18) to the Borel one, we have to integrate over all collective variables $x_I, x_A, \rho_I, \rho_A, U$, and, also, over x at fixed action (11), that is to say, at fixed value of the combination ξh. This problem may be divided into two parts. The integration over x and x_I is rather tedious algebraical problem due to the complicated form of the correlation function of quark currents in the instanton field (18),(19),(17). But from physical point of view, the main problem is the integration over size ρ_A and position $R = x_A - x_I$ of the large anti-instanton. There are two competing effects here. First, the factor $d(\rho_A) \sim \rho_A^b$ tends to make the integral over ρ_A divergent. Second, the almost zero fermion modes $\lambda^{2N_f} \sim \rho_I^{2N_f}/\rho_A^{4N_f}$ (14) tend to suppress the contribution of large anti-instantons. As a result, the value of the integral (18) (as well as the validity of our method) depends strongly on the number of light quarks N_f. The simple dimensional analysis (let us note, that the integrations over $d\rho_A$ and $d^4 x_A$ are not independent and $\rho_A \sim x_A - x_I$ owing to the constraint $\xi h = const$ (11)) shows that the critical value is $N_f = N_c$. If $N_f < N_c$ the first effect dominates and the integral (18) diverges at large ρ_A. Nevertheless, just in this case the well defined instanton induced asymptotics of perturbation theory may be extracted from (18). For calculation of the integral (18) beyond the perturbation theory at $N_f < N_c$ the new physical income is necessary (for example, one may consider the instanton liquid). The most favorable case is $N_f = N_c$. In this case, the integral over ρ_A in (18) diverges only logarithmically. As a result, we are able not only to obtain the asymptotics of perturbation theory, but also to calculate (at least with the logarithmic accuracy) the finite nonperturbative instanton–anti-instanton pair contribution to $R_{e^+e^-\to hadrons}$ and $R_{\tau\to hadrons}$.

If $N_f > N_c$, the attraction of pseudoparticles which appears owing to fermionic zero modes prevails. As a result, the integral is saturated by $\rho_A \sim R \sim \rho_I$ and the approximation of almost noninteracting pseudoparticles does not work. The problem of instanton induced asymptotics might also have the solution in this case, but, at least, this solution requires the considerable modification of the method used in the present paper.

With the use of (20), (21) let us extract the ρ_A dependent part from (18):

$$d(\rho_A) = \phi(\rho_I^2/\rho_A^2)\left(\frac{\rho_I}{\rho_A}\right)^b d(\rho) , \quad \phi(x) = \left[1 + b\frac{\alpha_s}{4\pi}\ln(x)\right]^{2N_c - \frac{b'}{2b}}\qquad (22)$$

Here and below $\alpha_s \equiv \alpha_s(q) \simeq \alpha_s(\rho_I)$. In order to find the leading perturbative asymptotics, one may assume $\phi(x) \equiv 1$ (as it was done in [11]), but

the calculation of the correlator (18) beyond the perturbation theory requires the use of the function $\phi(x)$ in the form (22). Moreover, the expansion of $\phi(x)$ in a series in $(\alpha_s \log(x))^k$ generates an important set of preasymptotic corrections $\sim (\log(n)/n)^k$ to the leading asymptotics.

The integral over ρ_A and $R = x_A - x_I$ for a fixed value of ξ gives:

$$\int \phi(\rho^2/\rho_A^2)\rho_A^{b-5}\delta\left(\frac{\rho^2\rho_A^2}{(R^2+\rho_A^2)^2}-\xi\right)d\rho_A d^4R = \frac{\pi^2}{2(b-2)(b-1)}\frac{\rho^b}{\xi^{b/2+1}}\phi(\xi).$$
(23)

Now we would like to present in more compact form the expressions for $T_{\mu\nu}$ and B_μ (18). Actually, the traces $T_{\mu\mu}$ and $B_\mu(x)B_\mu(y)$ are of interest because the polarization operator $\Pi_{\mu\nu}$ is transverse. As for B_μ, the situation is rather simple. Formally, the Green function at coincident points involved to B_μ goes to infinity. Nevertheless, the contribution to B_μ from the one-instanton Green function (17) has to be of the form:

$$Tr\{\gamma_\mu G_I(x,x)\} = (x-x_I)_\mu f(x-x_I)$$
(24)

because $(x-x_I)_\mu$ is the only possible vector. As a consequence of transverseness of B_μ: $\partial_\mu B_\mu = 0$, one immediately gets $f(x) \equiv 0$. Thus, the only nontrivial contribution to B_μ originates from zero modes:

$$B_\mu(x)B_\mu(y) = Tr\{\gamma_\mu S_\lambda(x,x)\}Tr\{\gamma_\mu S_\lambda(y,y)\} = \frac{(x,y)}{\pi^4 T_x^2 T_y^2}(H-2D),$$ (25)

$$H = \frac{TrO^+O}{|TrO|^2}, \quad D = Re\left(\frac{DetO}{[TrO]^2}\right).$$

Here $x_I = 0$.

After some tedious algebraic manipulations one gets:

$$T_{\mu\mu}(x,y) = \frac{1}{\pi^4 T_x^2 T_y^2}\left\{\frac{2\rho^4}{z^2} - \frac{4\rho^2(r,z)^2}{z^4} + \rho^2 - 2(x,y)D\right\}$$
(26)

$$-\frac{1}{\pi^4 z^4}\left\{\frac{(y,z)}{T_y^2} - \frac{(x,z)}{T_x^2}\right\} + \left(\begin{array}{c}\text{odd over } r_\mu \\ \text{terms}\end{array}\right),$$

where $z_\mu = (x-y)_\mu$, $r_\mu = \frac{1}{2}(x+y)_\mu$. Here the coordinates x and y are measured from the instanton center and hence, like it was done in the papers [12, 13], the integral over d^4x_I may be replaced by the integral over d^4r

$$\int T_{\mu\mu}(x,y)d^4r = \frac{1}{2\pi^2 z^2} + \frac{2}{\pi^2}\int_0^1 du\frac{u\bar{u}}{T^2}\left\{u\bar{u}\rho^2 - (T+\rho^2)D\right\},$$ (27)

$$\int B_\mu(x)B_\mu(y)d^4r = \frac{1}{\pi^2}\int_0^1 du\,u\bar{u}\frac{T+\rho^2}{T^2}(H-2D),$$

where $T = \rho^2 + u\bar{u}z^2$ and $\bar{u} = 1 - u$. Now it is easy to find the Fourier transforms:

$$\int e^{iqx}T_{\mu\mu}(x,0)d^4x d^4x_I =$$

$$\frac{2}{q^2} + 4\rho^2\int_0^1 du\left\{K_0\left(\frac{q\rho}{\sqrt{u\bar{u}}}\right) - \frac{1}{u\bar{u}}K_2\left(\frac{q\rho}{\sqrt{u\bar{u}}}\right)D\right\},$$ (28)

$$\int e^{iqx}B_\mu(x)B_\mu(0)d^4x d^4x_I = 2\rho^2\int_0^1 \frac{du}{u\bar{u}}K_2\left(\frac{q\rho}{\sqrt{u\bar{u}}}\right)(H-2D),$$

where K_0 and K_2 are the McDonald functions.

The last integral we are to calculate is the integral over the size of the "small" instanton $- \rho$. As we have seen above (6,23), the ρ_I-dependence of the integrand (except for that contained in (28)) in the large logarithms approximation arises as the multiplier $d^2(\rho)/\rho^5 \sim \rho^{2b-5}$. From the point of view of the Operator Product Expansion the large instanton and anti-instanton lead to the power correction in $\Pi_{\mu\nu}$

$$\frac{(q_\mu q_\nu - q^2\delta_{\mu\nu})}{q^4}\frac{\alpha_s}{6\pi}\sum_f e_f^2\int d^4x Tr\left(G_{\mu\nu}^{I+A}(x)G_{\mu\nu}^{I+A}(x)\right).$$
(29)

In our formulas this power correction corresponds to the term $2/q^2$ in the first integral (28). The integral over the instanton size for such correction to $R_{e+e^-\to hadrons}$ diverges like $\sim 1/q^4\int d\rho_I\rho_I^{2b-5}$. However, the $\sim 1/q^4$ corrections, which account for the long-wave vacuum fluctuations, are natural to be considered as the part of the infrared renormalon. Thus, we shall omit this term. All other terms in (28) contain the exponentially decreasing McDonald functions and, hence, the ρ-integral converges:

$$\int K_0\rho^{2b-3}d\rho du = \frac{2^{2b-4}}{q^{2b-2}}\frac{[(b-1)!(b-2)!]^2}{(2b-1)!},$$ (30)

$$\int K_2\rho^{2b-3}\frac{d\rho du}{u\bar{u}} = \frac{2^{2b-4}}{q^{2b-2}}\frac{(b-1)![(b-2)!]^2(b-3)!}{(2b-3)!}.$$

The coefficient $b \sim 10$ may be considered as the large parameter. Therefore, the both integrals (30) have the form $\int e^{-2q\rho}\rho^{2b}d\rho$ and are saturated by $\rho \approx b/q$. Thus, the size of the "small" instanton turns out to be b times larger than the typical wave length. In essence, both right hand sides of (30) are $\sim (b/q)^{2b}e^{-2b}$.

At last, collecting together (6), (18), (28) and (30), we get:

$$\Pi(q^2) = -\frac{\pi^2}{3} 4^{b+N_f-2} \frac{[(b-2)!(b-3)!]^2}{(2b-3)!} d^2(q) \times \tag{31}$$

$$\int dU\, d\xi\, |TrO|^{2N_f} \xi^{\frac{11}{6}(N_f-N_c)-1} \phi(\xi) \exp\left\{\frac{4\pi}{\alpha_s}\xi h\right\} A\,,$$

where (see (25))

$$A = \left(4D - \frac{2b-4}{2b-1}\right)\sum e_f^2 + (2H - 4D)(\sum e_f)^2\,. \tag{32}$$

This result coincides, up to misprints, with that obtained in [11] (of course for $\phi(\xi) = 1$).

The formula for $\Pi(q^2)$ may be rewritten in the form of Borel integral if one introduces the new variable $t = 1 - h\xi$:

$$\Pi(q^2) = Const \left(\frac{4\pi}{\alpha_s(q)}\right)^{4N_c} \int dU\, |TrO|^{2N_f} A|h|^{\frac{11}{6}(N_c-N_f)} \times \tag{33}$$

$$\left[\theta(h)\int_0^1 (1-t)^{\frac{11}{6}(N_f-N_c)-1} \exp\left\{-\frac{4\pi}{\alpha_s}t\right\}\phi\, dt + \right.$$

$$\left. \theta(-h)\int_1^\infty (t-1)^{\frac{11}{6}(N_f-N_c)-1} \exp\left\{-\frac{4\pi}{\alpha_s}t\right\}\phi\, dt\right]\,.$$

Here $\phi = \phi(|1-t|)$ (22). It will be recalled that formally $\phi = 1 + O(\alpha)$, but close to the singularity $t = 1$ taking into account the function ϕ changes completely the value of the integrand. θ-function in (33) takes the values 1 or 0 in accordance with the sign of its argument.

As we have already said, our method does not work at $N_f > N_c$. In fact, if $N_f > N_c$ the integral (33) over the orientations of the matrix U diverges at $h = 0$. The physical nature of the divergency is clear. We have used the Faddeev-Popov unity in order to fix the value of the instanton-anti-instanton interaction $S_{int} = \frac{4\pi}{\alpha_s}(t-1)$ (recall that our formulas are valid only for $|t-1| \ll 1$). However, if $N_f > N_c$ the main contribution to the integral comes from the configurations for which although the dipole-dipole interaction is formally small, the pseudoparticles themselves have the comparable sizes and strongly overlap. Of course, one can hardly find reliable result in this situation. In particular, our method does not work for such an attractive problem as the calculation of the perturbation theory asymptotics for $\Gamma_{Z_0\to hadrons}$ ($N_c = 3$, $N_f = 5$).

Of course, our formulas are valid if $N_f < N_c$. However, there are just 3 light quarks in nature and the case $N_f < N_c$ seems to be mostly of academic interest. Furthermore, for $N_f < N_c$ the correct result for $R_{e+e-\to hadrons}$ (up to trivial modification of single instanton density, see discussion after eq. (20)) has been obtained by Balitsky [11].

Therefore, we would like to consider in detail only the most interesting case $N_f = N_c$. If, in addition, $N_c = 3$ the formula (33) for $\Pi(q^2)$ takes the form:

$$\Pi(q^2) = -\frac{e^{8/3}[7!6!]^2}{3\pi^2 16!}\left(\frac{4\pi}{\alpha_s}\right)^{12}\left(0.510\int_0^1 dt\, \frac{\exp\left(-\frac{4\pi}{\alpha_s}t\right)}{1-t}\phi + \right. \tag{34}$$

$$\left. 0.054\int_1^\infty dt\, \frac{\exp\left(-\frac{4\pi}{\alpha_s}t\right)}{t-1}\phi\right)\,,$$

Here the averaging over $SU(3)$ group was performed numerically (see Appendix).

The expression (34) is enough to find the leading asymptotics of perturbative expansion for $\Pi(q)$ (here we put $\phi = 1$):

$$\Pi(q^2) = \sum \Pi_n \left(\frac{\alpha_s}{4\pi}\right)^n\,, \tag{35}$$

$$\Pi_n = -\frac{e^{8/3}}{3\pi^2}\frac{[7!6!]^2}{16!}0.510(n+11)! = -0.156(n+11)!\,.$$

We see that instanton-anti-instanton induced contribution to the perturbation theory series do has a huge enhancement $(n+11)!$.

Both integrals in (34) diverge at $t = 1$ (although the main physical problem is to interpret the contribution of small t to $\Pi(q^2)$ (34)). In the configuration space these divergences are related to the integration over almost noninteracting instanton and anti-instanton. Since the divergence is only logarithmic one can try out the physical intuition in order to restrict the range of integration in (34). The natural cut-off for ρ_A is $\rho_A \ll 1/\Lambda_{QCD}$, or, in terms of t

$$|t-1|_{min} \sim \left(\frac{\rho_I}{\rho_{Amax}}\right)^2 < \frac{\Lambda^2}{q^2}\,. \tag{36}$$

If so, the nonperturbative part of (34) may be found explicitly. To this end let us supplement the first term in (34) up to the principal value integral:

$$\Pi(q^2) \sim 0.510\,P\int_0^\infty dt\,\frac{\exp\left(-\frac{4\pi}{\alpha_s}t\right)}{1-t}\phi(|1-t|)+ \qquad (37)$$

$$\frac{48}{85}e^{-\frac{4\pi}{\alpha_s}}\int_{(\Lambda/q)^2}^\infty \frac{dx}{x}\phi(x)\exp\left(-\frac{4\pi}{\alpha_s}x\right).$$

By the use of the logarithmic variable $y = \frac{b}{4\pi}\ln(1/x)$ in the last integral one gets (up to correction $\sim \alpha_s$)

$$\int_{(\Lambda/q)^2}^\infty \frac{dx}{x}\phi(x)\exp\left(-\frac{4\pi}{\alpha_s}x\right) \approx \frac{4\pi}{b}\int_0^{1/\alpha_s}(1-\alpha_s y)^{22/9}dy = \frac{4\pi}{31\alpha_s}. \qquad (38)$$

Note, that if one replaces $\phi(x)$ by 1, the result for the nonperturbative correction will be 31/9 times larger. Finally, collecting together (37) and (38) one finds:

$$\Pi(q^2) = -\frac{e^{8/3}[7!6!]^2}{3\pi^2 16!}\left(\frac{4\pi}{\alpha_s}\right)^{12}\left(0.510\,P\int_0^\infty dt\,\frac{\exp\left(-\frac{4\pi}{\alpha_s}t\right)}{1-t}\phi(|1-t|)+ \qquad (39)\right.$$

$$\left.\frac{48}{2635}\frac{4\pi}{\alpha_s}e^{-\frac{4\pi}{\alpha_s}}\right).$$

Let us say a few words about the obtained nonperturbative correction $\sim e^{-\frac{4\pi}{\alpha_s}}$. Effectively, the integration over t in (33), (39) may be thought as the integration over the size of large anti-instanton. In the logarithmic scale $\ln(\rho_A\Lambda) \sim \alpha_s(\rho_A)^{-1}$ and one may say that the integration is performed over the (inverted) running coupling constant. The remarkable feature of our result is that all the values of $\alpha_s(\rho_A)$ in the whole range $\alpha_s(\rho_I) < \alpha_s(\rho_A) \ll 1$ make the comparable contribution to the nonperturbative part.

The formula (39) correctly accounts for the high orders of the perturbation theory and the nonperturbative corrections. However, the first terms of the perturbative expansion are wrong in (39). The only way to eliminate this defect is to calculate explicitly the first $3-5-10$ terms of the perturbation theory. After such modification the expression (39) takes the form:

$$\Pi(q^2) = \sum_0^N \Pi_{n\,exact}\left(\frac{\alpha_s}{4\pi}\right)^N - \frac{e^{8/3}}{3\pi^2}\frac{[7!6!]^2}{16!}\left(\frac{4\pi}{\alpha_s}\right)^{12}\times \qquad (40)$$

$$\left(0.510\,P\int_0^\infty dt\,\frac{t^{N+12}}{1-t}\exp\left(-\frac{4\pi}{\alpha_s}t\right) + 0.0182\frac{4\pi}{\alpha_s}e^{-\frac{4\pi}{\alpha_s}}\right).$$

Here, for simplicity, we put $\phi = 1$ in the principal value integral (39). The value of N should be large enough so that the perturbative series at $n = N$ will be consistent with the asymptotics (35). It is this issue that is most crucial for the practical applications of our method. As one can see, the asymptotic prediction (35) turns out to be many times larger than the "experimentally known" first $2-3$ terms of the perturbative series for β-function or $R_{e^+e^- \to hadrons}$. Of course, sooner or later, the series will reach the asymptotics (35). However, if it happens only at $n > 10$ (and it is presumably the case), the issue of the comparison with the "experiment" will be only of academic interest. Moreover at $n > 10$ the renormalons (1), (2) also may become important.

The only consistent way to find the extent of validity of the formula (35) is to consider the $1/n$ corrections to the leading asymptotics. The analogous calculation for the double well oscillator has been performed in our paper [15]. It is worth mentioning that the accuracy of the asymptotic formula for oscillator after taking into account of the $\sim 1/n$ corrections turns out to be not worse than 12% starting from $n = 5$.

Even for simple quantum-mechanics the calculation of the $1/n$ corrections to the instanton induced asymptotics requires considerable efforts. The corrections of the same order of magnitude arose while one takes into account the corrections to the instanton–anti-instanton interaction (both classical $\sim e^{-2T}/g^2$ and quantum $\sim e^{-T}$) and the two-loop correction $\sim g^2$ to the single instanton density. The complete calculation of the similar corrections in QCD seem does not possible. Nevertheless, the corrections may be found which have some additional enhancement. Firstly, these are the corrections of the order of $\sim \ln(n)/n$. In all formulas for the polarization operator $\Pi(q^2)$ we studiously kept the function $\phi(x)$ (22) which takes into account at the two-loop level the variation of the coupling constant $\alpha_s(\rho_A)$. As we have shown above the use of the function $\phi(x)$ is necessary for the correct calculation of the nonperturbative $\sim e^{-\frac{4\pi}{\alpha_s}}$ correction to $\Pi(q^2)$ (39). On the other hand, expanding $\phi(x)$ in the powers of α_s one gets the $\sim [\alpha_s \ln(1-t)]^k/(1-t)$ corrections to the integrand (39). Each next term of this series contains the small factor α_s but also the more strong singularity due to additional $\ln(1-t)$. It is easy to show that after integration over t each factor $[\alpha_s \ln(\frac{1}{1-t})]^k$ leads to $(\ln(n)/n)^k n!$. The asymptotic formula (35) with all the $\sim (\ln(n)/n)^k$ included takes the form:

$$\Pi_n = -0.156 \left(1 - \frac{3N_c \ln(n)}{n}\right)^{\frac{5}{6}N_c - \frac{1}{6N_c}} (n+11)! \qquad (41)$$

$$= -0.156 \, (n)^{-\frac{5N_c^2-1}{2n}} (n+11)! \ .$$

As we have said in the introduction, the asymptotic series (35) at very large n behave as $n^{11} n^n$. In (41) we have shown explicitly that taking into account of the asymptotic freedom for the large anti-instanton ($\phi(x)$) leads to additional factor $n^{\frac{1}{n}}$.

We wrote explicitly the factor $5N_c^2$ in the argument of the exponent in (41) in order to show that, unfortunately, the correction to the leading asymptotics is of the order of $N_c^2/n \sim 10/n$. We may also search for the corrections of the order of $\sim N_c^2/n$ enhanced by the additional "large" factor $\ln(b) = \ln(3N_c)$. The correction $\sim N_c^2 \ln(n)/n$ (41) arose after the function $\alpha_s(\rho_A)$ was treated consistently at large ρ_A. However, (see discussion after formula (30)), the size of the "small" instanton also is parametrically large compared to the characteristic wave length $\rho_I \sim b/q$. Taking into account this effect leads to the additional factor in (31):

$$\left(\frac{\ln\left(\frac{1}{\rho_I \Lambda_{QCD}}\right)}{\ln\left(\frac{q}{\Lambda_{QCD}}\right)}\right)^{4N_c - b'/b} = \left(1 - \frac{\alpha_s(p)}{4\pi} 2b \ln(b)\right)^{4N_c - b'/b} . \qquad (42)$$

Finally, we have:

$$\Pi_n = -0.156 \left((3N_c)^4 n\right)^{-\frac{5N_c^2-1}{2n}} (n+11)! \ . \qquad (43)$$

Of course, this result is valid only at $n \gg N_c^2$. Nevertheless it may happen that $n \sim N_c^2 \ln(3N_c)$.

Some other $\sim 1/n$ corrections also are easy to find. For example, one may calculate the correction induced by the quantum part of the instanton–anti-instanton interaction. In fact, in the leading approximation this correction will be taken into account if one replaces in the dipole-dipole interaction $S_{int} = -\frac{4\pi}{\alpha_s} \xi h$ (11) the running constant $\alpha_s(q)$ by the $\alpha_s(\sqrt{\rho_I \rho_A})$.

However, the complete calculation of all $\sim 1/n$ corrections is too difficult, and not so actual. As we see the corrections to the leading asymptotics behave like N_c^2/n and, thus, the issue of quantitative comparison of the asymptotic prediction with the exactly calculated terms of the perturbative expansion will be possible only at $n > N_c^2$.

4 $R_{e^+e^- \to hadrons}$

Now we are able to calculate the instanton contribution to the cross section of the electron-positron pair annihilation into hadrons. We have already shown the well known formula (6) connecting $R_{e^+e^- \to hadrons}$ and imaginary part of the polarization operator. The main q^2-dependence of Π comes from the factor $\exp(-\frac{4\pi}{\alpha_s}) \sim q^{-2b}$. However, at integer b (in particular if $N_f = N_c$) the q^{-2b} term does not give rise to the imaginary part. The imaginary part does appear after the more gentle dependence is taken into account $\Pi(q^2) \sim [\ln(\frac{q^2}{\Lambda^2})]^\gamma q^{-2b}$. Let us note that because the imaginary part of the large logarithm $Im \ln(-q^2) = \pi$ is of the order of one, the $R_{e^+e^- \to hadrons}$ at $N_f = N_c$ acquires an additional power of α_s compared to the polarization operator $\Pi_{\mu\nu}$. After analytical continuation of the expression (39) one gets:

$$R_{e^+e^- \to hadrons}(q^2) = \qquad (44)$$

$$-11 e^{8/3} \frac{[7! 6!]^2}{15!} \left(\frac{4\pi}{\alpha_s}\right)^{11} \left(0.510 P \int_0^\infty dt \frac{\exp\left(-\frac{4\pi}{\alpha_s} t\right)}{1-t} \psi(|1-t|) + \frac{636}{28985} \frac{4\pi}{\alpha_s} \exp\left(-\frac{4\pi}{\alpha_s}\right)\right) ,$$

where

$$\psi(x) = \left(1 + \frac{9}{2} \frac{\alpha_s}{4\pi} \ln(x)\right) \left(1 + 9 \frac{\alpha_s}{4\pi} \ln(x)\right)^{\frac{13}{9}} . \qquad (45)$$

Also, it is easy to find the asymptotics of the perturbation theory together with the $\sim 1/n$ corrections discussed in the end of preceding section:

$$R_{e^+e^- \to hadrons}(q^2) = \sum R_n \left(\frac{\alpha_s}{4\pi}\right)^n , \qquad (46)$$

$$R_n = -11 e^{8/3} \frac{[7! 6!]^2}{15!} 0.510 \left(9^4 n\right)^{-\frac{35}{2n}} (n+10)! \approx$$

$$-813 \, (6561 n)^{-\frac{35}{2n}} (n+10)! .$$

As we have already said, the "experimentally" known first terms of the perturbation theory series R_n are in many orders of magnitude smaller than the naive asymptotics $(n+4N_c)!$. As well as for the Π_n (43) the $\sim 1/n$ corrections to R_n (46) tend to decrease the asymptotic prediction. Furthermore, if one extends the expression (46) down to $n \sim 1$, this suppression may even compensate the huge factor 10!. Let us remind, however, that one may believe

the formula (46) only at $n > 10$ (or even at $n \gg 10$). Moreover, deriving the formulas (43,46) we have assumed the following hierarchy of the $\sim 1/n$ corrections: $N_c^2 \ln(n)/n \gg N_c^2 \ln(N_c)/n \gg N_c^2/n$. But actually, as it is seen from (46), even at $N_c = 3$ the corrections $\sim N_c^2 \ln(N_c)/n$ are much more important than the corrections $\sim N_c^2 \ln(n)/n$ for any reasonable n. In such situation one also may expect a surprise from unknown $\sim N_c^2/n$ corrections.

Up to now we have not considered the question at what energies our asymptotic formulas may be used. Firstly, we calculate the functional integral by the steepest descent method. Hence, at least the coupling constant $\alpha_s(\rho)$ should be small or, in other words, $\rho \ll 1/\Lambda$. Moreover we consider only the case $N_f = N_c$, and so, the minimal value of ρ is determined by the mass of c quark $\rho_{min} \sim \frac{1}{m_c}$. On the other hand, as we have shown above, the effective inverse size of the "small" instanton turns out to be sufficiently smaller than the external momentum $q \sim b/\rho_I$. Hence, it is natural to expect that our asymptotic formulas are valid within the energy region:

$$3 Gev \ll qc < 15 Gev \ . \qquad (47)$$

Everywhere above by the coupling constant $\alpha_s(q)$ was meant the three-flavors coupling $\alpha_3(q)$. However, at q larger than $5 Gev$ it will be natural to express the result in terms of $\alpha_5(q)$. The relation between $\alpha_3(q)$ and $\alpha_5(q)$ in the leading approximation reads:

$$\frac{4\pi}{\alpha_3(q)} = \frac{4\pi}{\alpha_5(q)} + \frac{4}{3} \ln \left(\frac{q^2}{m_c m_b} \right) + O(1) \ , \qquad (48)$$

where m_c and m_b are masses of c and b quarks. As a result, instead of formula (44) we get

$$R_{e+e- \to hadrons}(q^2) = Const \left(\frac{m_c m_b}{q^2} \right)^{4/3} \left(\frac{4\pi}{\alpha_5} \right)^{11} P \int_0^\infty dt \frac{\exp\left(-\frac{4\pi}{\alpha_s}t\right)}{1-t} \qquad (49)$$

Here we do not account for any $\sim 1/n$ corrections and also do not write very small at $qc \sim 5 Gev$ nonperturbative correction $\sim 1/q^{18}$.

5 The calculation of the τ decay width

During the last few years a question about the possibility to extract the value of the strong coupling $\alpha_s(m_\tau)$ from the hadronic width of the τ-lepton has been actively discussed [21]-[24]. It has been shown that the "standard",

obtained in the framework of QCD sum rules nonperturbative corrections to the τ decay width, do not exceed a few percents [21]. The assumption that only first terms $\sim 1/q^n$ are important allows one to fix the value of the strong coupling at the m_τ pole within 10% accuracy.

The first attempt to go beyond the "standard" sum rules was made in the paper of M.Porrati and P.Nason [25]. However, the obtained result seems to be of only a methodical interest. The authors of [25] have found the single-instanton contribution in an empty (perturbative) vacuum and consequently obtained a completely negligible result proportional to the product of the light quark masses $m_u m_d m_s$.

Much more reasonable approach to the calculation of the instanton contribution to $R_{\tau \to hadrons}$ has been demonstrated in the work of I.Balitsky et al. [26]. They have tryed to take into account the influence on the instanton of the long-wave nonperturbative vacuum fluctuations. Roughly speaking, in the paper [26] the current quark masses were substituted by the effective masses [27] $m_q \to m_q - \frac{2}{3}\pi^2 \langle \bar{q}q \rangle \rho_I^2$. Therewith the instanton contribution turns out to be comparable with the "standard" nonperturbative contributions. However, the authors of [26] still do not put in doubt the possibility to extract the value of α_s from $R_{\tau \to hadrons}$.

In this section we would like to consider the contribution of the instanton-anti-instanton pair to the τ decay width. By considering the concrete topologically trivial configuration, we, unlike the authors of the papers [25, 26], are able to find the asymptotics of the perturbation theory. As for the nonperturbative contribution, at first sight one may expect that our "large" anti-instanton is only one of the examples of the long-wave vacuum fluctuations which give rise to the quark condensate $\langle \bar{q}q \rangle$ and, hence, our nonperturbative correction has been already included in the result of the work [26]. It would be actually the case if our universe was built only from one or two light quarks ($N_f < N_c$). In this case the integral over the size of the large anti-instanton for the polarization operator (18,33) diverges and it would be natural to evaluate it via some phenomenological quark condensate $\langle \bar{q}q \rangle \sim \Lambda_{QCD}^3$. However, as we have seen in two preceding sections, in the case $N_f = N_c$ the integral over ρ_A with the logarithmic accuracy comes from the whole region $1/q \ll \rho_A \ll 1/\Lambda_{QCD}$. In particular, it means that we have found explicitly the most probable long-wave background for the small instanton. As we will see bellow, the instanton–anti-instanton pair induced correction to $R_{\tau \to hadrons}$ turns out to be much larger than the semi-phenomenological result of the paper [26].

The reliability of our result also requires the separate consideration. The typical size of "small" instanton turns out to be $b \sim 10$ times larger than

$1/q$. Applied to the τ decay it means that the typical ρ is of the order of $b/m_\tau \sim 1/\Lambda_{QCD}$ to say nothing about the "larger" anti-instanton. In such situation we may obtain only more or less reliable estimate of the effect on the order of magnitude.

Like $R_{e^+e^- \to hadrons}$, the ratio of hadronic τ decay width to its leptonic width $R_{\tau \to hadrons}$ may be found by the analytical continuation of the correlator of the weak currents from the euclidean q^2 region (see e.g. [25, 26]):

$$R_{\tau \to hadrons} = -6i\pi \oint_{|s|=1} ds(1-s)^2[(1+2s)\Pi^T(-sm_\tau^2) + \Pi^L(-sm_\tau^2)], \quad (50)$$

$$\Pi_{\mu\nu}(q^2) = \Pi^T(q^2)(q_\mu q_\nu - q^2\delta_{\mu\nu}) + \Pi^L(q^2)q_\mu q_\nu = \int dx e^{iqx}\langle j_\mu^+(x)j_\nu(0)\rangle ,$$

where

$$j_\mu = V_{ud}u^+\gamma_\mu(1+\gamma^5)d + V_{us}u^+\gamma_\mu(1+\gamma^5)s . \quad (51)$$

Here the first integral is taken from upper to lower edge of the cut which goes along the positive real axis of the complex s-plane. In order to calculate the correlator (50), one may use the formulas of the preceding sections with minimal modifications. The first distinction is that the weak current (51) include the product of two quark operators with different flavors. Hence, the non-connected graphs which in the case of the correlator of two electro-magnetic currents (18) led to the term proportional to $\left(\sum e_f\right)^2$, do not appear in (50) (note, in the case of electro-magnetic currents the term $\sim \left(\sum e_f\right)^2$ vanishes at $N_f = N_c = 3$ due to vanishing of the sum of quark charges). The fermionic Green function (16),(17) anticommute with γ^5. As a result, the vector-vector and axial-axial contributions to the correlator (50) are equal. At last, one has to make the substitution $\sum_f e_f^2(= 2/3) \to (|V_{ud}|^2 + |V_{us}|^2)(\simeq 1)$ while going from (18) to (50). Note that both correlators (18) and (50) turn out to be transverse. Thus, we have $\Pi^T(q^2) = 3\Pi(q^2)$ (see (39)) and $\Pi^L(q^2) = 0$. Now, integrating over s, one can easily find $R_{\tau \to hadrons}$ in the leading over α_s approximation

$$R_{\tau \to hadrons} = \frac{33e^{8/3}}{40}\frac{[7!6!]^2}{15!}\left(\frac{4\pi}{\alpha_s}\right)^{11} \times \quad (52)$$

$$\left(0.510 P\int_0^\infty dt\frac{\exp\left(-\frac{4\pi}{\alpha_s}t\right)}{1-t}\psi(|1-t|) + \frac{636}{28985}\frac{4\pi}{\alpha_s}e^{-\frac{4\pi}{\alpha_s}}\right) .$$

This expression leads to the following asymptotics of the perturbation theory:

$$R_{\tau \to hadrons} = \sum R_{n\tau}\left(\frac{\alpha_s}{4\pi}\right)^n \quad (53)$$

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$$R_{n\tau} = 61(n+10)!\left(9^4 n\right)^{-\frac{35}{2n}} .$$

Just as for $R_{e^+e^- \to hadrons}$ (46), there appear certain problems while interpreting the instanton–anti-instanton pair induced asymptotics of the perturbation theory for $R_{\tau \to hadrons}$. In fact, the asymptotics (53) is valid only at $n \gg 10$, where the renormalon contributions (1),(2) are much larger than the instanton one. We may only hope that at $n \sim 10$ formula (53) does give the correct estimate of the instanton contribution on the order of magnitude. If so, at $n \sim 10$ the instantons will dominate over renormalons. In the most interesting case $n = 3 - 5$ our formulas do not work.

In order to compare our result with the experiment, let us consider in more detail the pure nonperturbative $\sim e^{-\frac{4\pi}{\alpha}}$ term in (52) (note that this correction in terms of Λ_{QCD} (21) behaves like $(\Lambda/m_\tau)^{18}$). The quantities R_τ^{np} for popular values of $\alpha_s(m_\tau)$ are shown in the first column of the Table. One has to compare these values with the experimental value $R_{\tau \to hadrons} = 3.56 \pm 0.03$ [23] (moreover, the last quantity contains a large trivial part $R_{\tau \to hadrons} \approx N_c$ and one has to extract the value α_s only from the remainder $R_{\tau \to hadrons} - 3 = 0.56 \pm 0.03$). As we can see, the nonperturbative correction turns out to be dramatically large.

There exist, however, the procedure (used also in the work [26]) which allows to reduce the huge discrepancy with the experiment. The regular way to improve the nonperturbative correction (52) is to calculate the $\sim \alpha_s$ corrections to it. Since, as will be shown bellow, the result is going to be deceased by $30 - 50$ times, one has to sum an infinite series of the corrections. Undoubtedly, the exact calculation even of the first correction of the order of α_s to (52) is beyond our abilities. All we can do is to use the dependence of the coupling constant on the instanton size $\alpha_s(\rho)$, which is known from the renorm-invariance principle. Rather weak justification for taking into account just these particular corrections is the fact that they are enhanced as $\ln(3N_c)$ compared to the other ones. As we have said above, the typical inverse size of the "small" instanton is much smaller than the external momentum $(\rho^*)^{-1} \sim m_\tau/b$. The dependence of the coupling constant on the instanton size in the leading one-loop approximation has been already taken into account when we wrote the instanton density in the form $d(\rho) \sim \rho^b$. At first sight, in order to account for the two-loop effects, one has simply to add to the nonperturbative contribution (52) the factor (see (42)) :

$$\left(\frac{4\pi}{\alpha_s}\right)^{13}e^{-\frac{4\pi}{\alpha_s}} \to \left(\frac{\ln\left(\frac{m_\tau}{b\Lambda}\right)}{\ln\left(\frac{m_\tau}{\Lambda}\right)}\right)^{\frac{53}{9}}\left(\frac{4\pi}{\alpha_s}\right)^{13}e^{-\frac{4\pi}{\alpha_s}} . \quad (54)$$

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Unfortunately, just for τ-lepton the formula (54) turns out to be very unstable. Deriving (54), we have implied that $\ln\left(\frac{m_\tau}{b\Lambda}\right) \gg 1$, while actually $m_\tau \approx b\Lambda$. In order to improve the situation, we will try to insert the exact function $[-\ln(\rho\Lambda)]^{\frac{53}{9}}$ in the ρ-integral (30), as it was done in the paper [26]. While doing so, we effectively shift the maximum of the integrand from the dangerous region $\rho \sim 1/\Lambda$ for $53/9 \approx 6$ is sufficiently large quantity. As the result, the main contribution to $R^{np}_{\tau \to hadrons}$ comes from the instantons of the size $\rho \sim 4/m_\tau$ rather than $\rho \sim b/m_\tau$. Integrating over the Borel parameter (see (34)-(39)) before the ρ-integration (just after the Borel integration the full multiplier $[\ln(\rho\Lambda)]^{\frac{53}{9}}$ is gathered), we obtain the following expression for the nonperturbative correction to $\Pi(q^2)$:

$$\Pi^{np}(-sm_\tau^2) = \frac{e^{8/3}}{\pi^2 \cdot 7 \cdot 9 \cdot 31 \cdot 2^{18}} \left(\frac{4\pi}{\alpha_s(m_\tau)}\right)^{13} \exp\left(-\frac{4\pi}{\alpha_s(m_\tau)}\right) \times \quad (55)$$

$$\int_0^1 du \int \frac{dx}{(-s)^9} x^{15} \left\{ \frac{1}{2u\bar{u}} K_2\left(\frac{x}{\sqrt{u\bar{u}}}\right) - \frac{7}{5} K_0\left(\frac{x}{\sqrt{u\bar{u}}}\right) \right\} \left(1 + \frac{9\alpha_s}{4\pi}\ln\left(\frac{-s}{x^2}\right)\right)^{\frac{53}{9}}.$$

Here we have also performed the analytical continuation (50) $m_\tau^2 \to -sm_\tau^2$. In order to calculate the integral over x, one may use the identity:

$$\ln(p)^n = \lim_{\varepsilon \to 0} \left(\frac{\partial}{\partial \varepsilon}\right)^n p^\varepsilon. \quad (56)$$

Now, with the use of (50), we can find the nonperturbative correction to the τ decay width

$$R^{*np}_{\tau \to hadrons} = \lim_{\varepsilon \to 0} -i \frac{e^{8/3}}{\pi \cdot 7 \cdot 31 \cdot 2^{17}} \left(\frac{4\pi}{\alpha_s(m_\tau)}\right)^{13} \exp\left(-\frac{4\pi}{\alpha_s(m_\tau)}\right) \times \quad (57)$$

$$\left(1 + \frac{9\alpha_s}{4\pi}\frac{\partial}{\partial \varepsilon}\right)^{\frac{53}{9}} \int ds \frac{1 - 3s^2 + 2s^3}{(-s)^{9-\varepsilon}} dx\, x^{15-2\varepsilon} du \left\{\frac{K_2}{2u\bar{u}} - \frac{7}{5}K_0\right\}.$$

After trivial integration over x, s and u one gets:

$$R^{*np}_{\tau \to hadrons} = \lim_{\varepsilon \to 0} \frac{9e^{8/3}}{\pi \cdot 70 \cdot 31} \left(\frac{4\pi}{\alpha_s(m_\tau)}\right)^{13} \exp\left(-\frac{4\pi}{\alpha_s(m_\tau)}\right) \times \quad (58)$$

$$\left(1 + \frac{9\alpha_s}{4\pi}\frac{\partial}{\partial \varepsilon}\right)^{\frac{53}{9}} \sin(\varepsilon\pi) \frac{(12-\varepsilon)\Gamma(9-\varepsilon)\Gamma(8-\varepsilon)^2\Gamma(5-\varepsilon)}{2^{2\varepsilon}\Gamma(18-2\varepsilon)}.$$

This expression also may be presented as a series in powers of α_s. Note, that the first term of the series, corresponding to $(\partial/\partial\varepsilon)^0$, is vanishing. This leads to loss of one factor $1/\alpha_s(m_\tau)$ in $R_{\tau \to hadrons}$ (similarly, as in the case of $R_{e+e- \to hadrons}$) compared to the correlation function. Coefficients of the expansion of (58) in powers of α_s were found numerically. The resulting quantities $R^{*np}_{\tau \to hadrons}$ are shown in the last column of the Table. The terms of series in α_s, generated by $\left(1 + \frac{9\alpha_s}{4\pi}\frac{\partial}{\partial\varepsilon}\right)^{\frac{53}{9}}$, are maximal at $n \sim 2 \div 3$ and become negligible at $n > 8$. As we can see, taking into account of the two-loop dependence of $\alpha_s(\rho)$ allows to reduce the nonperturbative correction almost by two orders of magnitude. Nevertheless, the formula (58) turns out to be surprisingly stable. We have used also the Stirling formula for approximate calculation of the Γ-functions in (58) and the approximate values of $R^{*np}_{\tau \to hadrons}$ found in this way almost coincide with results of numerical calculations listed in the table.

It is seen from the table that passing from $\alpha_s = 0.28$ to $\alpha_s = 0.29$ our correction changes the sign. Moreover, the ratio R^*/R in the interval $0.15 < \alpha_s < 0.43$ changes the sign three times, being everywhere smaller than $1/30$. Compared to the simple formula (54) such rich behaviour provides us one more evidence that effectively we work at very low energies and listed in the table result is, at best, only the estimate on the order of magnitude.

Table

$\alpha_s(m_\tau)$	$R^{np}_{\tau \to hadrons}$	$R^{*np}_{\tau \to hadrons}$
0.28	5.65	-0.0229
0.29	17.44	0.0507
0.30	49.23	0.476
0.32	311.1	6.70
0.34	1514.1	44.79
0.36	5943.4	190.1

The full nonperturbative contribution of the instanton–anti-instanton pair to the hadronic τ decay width $R^{*np}_{\tau \to hadrons}$ and the contribution accounting for only the leading over α_s term in the sum (58) $R^{np}_{\tau \to hadrons}$ at various values of $\alpha_s(m_\tau)$.

One should remember that the obtained results should be compared with the experimental value $(R_{\tau \to hadrons} - 3)_{exp} = 0.56 \pm 0.03$. We see that, in spite of all our effort, even for $\alpha_s = 0.29$ the instanton contribution to $R_{\tau \to hadrons}$ still remains large. In this situation the only way out may be to ignore completely the instanton contribution to $R_{\tau \to hadrons}$. For example, one may say that the series of the power corrections $\sim 1/m_\tau^n$ is also asymp-

totic and it is natural to cut off it somewhere at $n \sim 4 - 8$ (the instanton contribution behaves like $\sim 1/m_\tau^{18}$). Nevertheless, we do not know to what extent this point of view is justified and we consider our result as indication of the *impossibility* to extract the reliable value of α_s from $R_{\tau \to hadrons}$.

6   Appendix. Integration over the $SU(3)$ group

The average values of expressions polynomial in the elements of the $SU(N)$ matrix may be easily found by, for example, graphical method described in [29]. However, for calculation of the integrals including θ-functions like $\int |TrO|^6 \theta(h)$ (33) one has to specify in explicit form the parametrization of the matrix and the measure of integration. We use the following parametrization of the $SU(3)$ group element:

$$U = \qquad (59)$$

$$
\begin{pmatrix}
-P_2^* P_4^* s_1 s_2 - P_1 P_3 P_5^* c_1 c_2 s_3 & P_1^* P_4^* c_1 s_2 - P_2 P_3 P_5^* s_1 c_2 s_3 & P_3 c_2 s_3 \\
P_2^* P_3^* s_1 c_2 - P_1 P_4 P_5^* c_1 s_2 c_3 & -P_1^* P_3^* c_1 c_2 - P_2 P_4 P_5^* s_1 s_2 c_3 & P_4 s_2 s_3 \\
P_1 c_1 s_3 & P_2 s_1 s_3 & P_5 c_3
\end{pmatrix},
$$

where: $c_{1,2} = \cos \frac{\psi_{1,2}}{2}$, $s_{1,2} = \sin \frac{\psi_{1,2}}{2}$, $c_3 = \cos \psi_3$ and $s_3 = \sin \psi_3$, $P_n = e^{i\varphi_n}$. The parameters ψ_m and φ_n vary in the range: $0 < \psi_{1,2} < \pi$, $0 < \psi_3 < \pi/2$ and $0 < \varphi_n < 2\pi$. It may be shown that the matrix (60) is unitary and unimodular. The measure of integration over the matrix group U is given by:

$$dU = N \int \Pi \alpha_i |\det(M)|^{1/2}, \qquad (60)$$

$$M_{ij} = Tr(U^{-1}(\partial_i U)U^{-1}(\partial_j U)).$$

In our case the parameters α_i ($i = 1, ..., 8$) are the three angles ψ_m and five angles φ_n. After rather tedious algebra one obtaines:

$$dU = N \Pi d\psi_m \Pi d\varphi_n \sin \psi_1 \sin \psi_2 \sin^3 \psi_3 \cos \psi_3. \qquad (61)$$

The results of the averaging with this measure of the expressions $\langle |TrO|^6 \rangle$ and $\langle |TrO|^2 Re[detO(TrO^+)^2] \rangle$ coincide with those obtained by the graphical method [29]. The integration of the expressions including θ-functions has been performed numerically. The results are given bellow:

$$\langle |TrO|^6 \rangle = \frac{7}{5} \quad ; \quad \langle |TrO|^6 \theta(h) \rangle = 1.368 ;$$

$$\langle |TrO|^2 Re[detO(TrO^+)^2] \rangle = \frac{1}{2} \quad ; \quad \langle |TrO|^2 Re[detO(TrO^+)^2] \theta(h) \rangle = 0.473 ;$$

$$\langle A|TrO|^6 \rangle = \frac{48}{85} \quad ; \quad \langle A|TrO|^6 \theta(h) \rangle = 0.510. \qquad (62)$$

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**Instanton–anti-instanton pair
induced contributions
to $R_{e^+e^- \to hadrons}$ and $R_{\tau \to hadrons}$**

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**Индуцированные парой
инстантон-антиинстантон вклады
в $R_{e^+e^- \to адроны}$ и $R_{\tau \to адроны}$**