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ANOMALOUS DIFFUSION IN MICROTRON AND THE CRITICAL STRUCTURE ON THE CHAOS BORDER

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Anomalous diffusion in microtron and the critical structure on the chaos border

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Abstract

The results of numerical experiments and the theoretical analysis of the anomalous diffusion in the critical structure on the chaos–order as well as chaos–chaos border are presented. In the former case the critical exponent $c_D \approx 1/3$, which determines the anomalous diffusion rate $D \sim t^{c_D}$, as well as the correlation exponent $c_A \approx 1/2$ have been found to a good accuracy and in agreement with the prediction of the resonance theory of critical phenomena in dynamical systems. The most important result is confirmation of the basic conception in this theory on supercriticality of the local order parameter in a close vicinity of chaos border in the chaotic motion component.

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1 Introduction

Microtron was the first cyclic accelerator for relativistic particles (electrons) invented by Veksler [1]. The dynamics of energy gain in the microtron can be approximately descibed by a simple map $x,\, p \to \overline{x},\, \overline{p}$ over a period of electron's rotation in the magnetic field:

$$\overline{p} = p + K \cdot \sin x \,, \quad \overline{x} = x + \overline{p} \tag{1}$$

Here x is the phase of accelerating voltage with amplitude V_0 and frequency Ω. Canonically conjugated action p and the only model parameter K are related to electron energy E and maximal Larmor frequency ω_B as follows (in units $e = m = c = 1$):

$$|p| = \frac{2\pi E\Omega}{\omega_B}, \quad K = \frac{2\pi V_0 \Omega}{\omega_B} \tag{2}$$

Dynamics of microtron model (1) was studied in [1,2] and in many other papers (see, e.g., [3]). In all these studies the main attention was always paid to the regular acceleration ($|p| \propto t$, where t is the number of iterations for map (1)) which corresponds to the (neutrally) stable dynamics of x phase (undamping oscillation). That, microtron, acceleration regime is only possible for special values of parameter $K = K_n \approx 2\pi n$ where $n \neq 0$ is any integer. The stability domain size on phase plane (x, p) is very small and rapidly decreases with n. Even the main domain $|n| = 1$ occupies less than 1% of the phase plane. What is going on for the rest of initial conditions?

How strange it may seem this question was addressed (and answered) much later when Veksler's model (1), due to its simplicity, became a basic one in the studies of nonlinear dynamics and chaos (see, e.g., [4,5]). This model is also called the standard map since many other physical problems can be reduced to such a map.

It was found that a considerable part of the phase plane corresponds to unbounded diffusion ($|p| \propto \sqrt{t}$) if $K > 1$, the latter apparently occupying the whole plane as K grows. The microtron turns into a "stochatron", the term introduced in [6] where a diffusion accereration has been proposed using some noise voltage. Instead, it was sufficient to simply change the motion initial conditions (a little) and/or the parameter K (in a wide range) [7]. To my knowledge, neither of these suggestions was ever realized or even experimentally attempted. However, in a different version (without microtron regimes) the dynamical chaos was used for an initial plasma heating in stellarator [8].

"Simple" model (1) proved to be very rich and was studied (and still is studying) both theoretically and in numerical experiments. It turned out that the motion statistical properties, particularly diffusion, may happen to be rather unusual, or "anomalous" (see, e.g., [9]). As was found this is related to the chaos border in the phase space where a very intricate hierarchical structure of motion arises. Even though the structure itself was studied in detail [10,9] its impact on the motion statistical properties remains essentially unclear [11,9]. The present paper is devoted just to this very problem.

2 Stability islets

The main domains of regular accelerations ("islets") on the phase plane of model (1) arise around fixed points (periodic solutions) $p = 0 \; mod \; 2\pi$, $x = \pm x_0$ where

$$K \cdot \sin x_0 = 2\pi n \,, \quad K^2 = s^2 + (2\pi n)^2 \,, \quad s = K \cdot \cos x_0 \,, \quad -4 < s < 0 \tag{3}$$

and the latter inequalities determine the stability region around a fixed point. In what follows $s = -2$ is asumed (the center of stability). For each $|n|$ there are two islets per phase space bin $2\pi \times 2\pi$. All the islets are similar in the dimensionless variables

$$x_s = \frac{x - x_0}{x_b} \,, \quad K \cdot x_b \approx 0.99 \,, \quad p_s = \frac{p}{p_b} \,, \quad K \cdot p_b \approx 2.49 \,, \tag{4}$$

In Fig.1 the borders of 5 islets are shown for $n = 2, 20, 200, 2000,\ 20000$. Inside the border the motion is regular, and the relative area A of this domain satisfies the relation

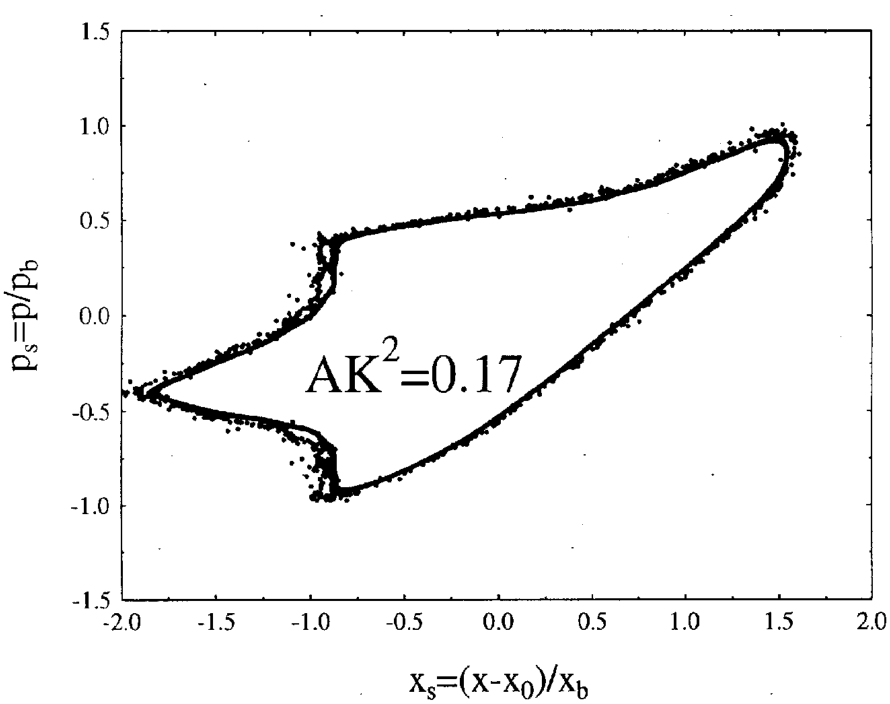

Figure 1: Universal border of microtron domains in dimensionless variables (4): $n = 2 - 20000$; motion time for each n is 3000 iterations of map (1). The motion within the border is regular, and that outside is chaotic. Near fixed point $x_s = p_s = 0$ the small oscillation frequency $\omega_0 = \pi/2$ $(r_0 = 1/4)$ while at the border $\omega_b = 2\pi r_b$ (6).

$$A \cdot K^2 \approx 0.17 \tag{5}$$

The maximal $A \cdot K^2 \approx 0.19$ is reached for the stability parameter $s \approx -1.92$. All scalings (3–5) are inferred from the theory [4,9,11]. However, the numerical factors are empirical. Islet's stability border determines the chaos–order transition, and it is robust (structurally stable) that is it persists under a small change of the only parameter K which is also the order parameter.

A peculiarity of the model under consideration is a very small size of the border as well as of the regular domain. Nevertheless, such a tiny border considerably modifies the statistical properties of the whole chaotic component as a result of trajectory "sticking" inside the complicated critical structure along the chaos border [11,9]. The structure is completely determined by the rotation number on the border

$$r = r_b = 0.23889... = [4, 5, 2, 1, 1, 1, 2, ...] \tag{6}$$

which is also n–independent. The continued–fraction representation of r is given by the latter expression (6) (the integers in parentheses are successive elements of a continued fraction). As the rotation number $r = \omega/2\pi$ is the ratio of oscillation frequency to that of perturbation (2π) such a representation does naturally single out the basic nonlinear resonances around the border which determine the critical structure. These resonances are related to the covergents $r_m = p_m/q_m \to r, m \to \infty$, each denominator q_m being equal to the motion period in the resonance.

A graphical picture of the critical structure and describing it renormalization group is given by the border motion spectrum presented in Fig.2a. It is the spectrum of radial oscillation $\rho(t)$, $\rho^2 = x_s^2 + p_s^2$, transverse to the chaos border. A characteristic feature of the spectrum is irregularity of the main peaks marked by m values. The periods of the corresponding resonances are: $q_m = 4, 17, 21, 38, 59, 97,\ 350, 447...$, $m = 1, 2, 3, 4, 5, 6, 7, 8....$ Such a picture is typical for the critical structure, and it is described by a chaotic renormgroup [9,11]. It means that the variation of the motion structure from one scale to the next is irregular being itself described by some statistical law (dotted line in Fig.2a). Odd resonances ($m = 1, 3, 5, 7...$) are situated within the stability domain (inside the chaos border) while even ones encircle the border that is fall into the motion chaotic component.

For comparison, the special case of exact scaling (a fixed point of the renormgroup) [10] is also shown in Fig.2b. Here the scale–to–scale transitions are perfectly regular. Curiously, the exactly regular scaling includes both regular as well as chaotic components (trajectories). In both cases in Fig.2 the border motion is almost periodic (of discrete spectrum), a finite peak width $\Delta\nu \sim 1/T$ relating to the total motion time T.

3 Critical structure and anomalous diffusion

The main level of the critical structure is determined by the series of basic nonlinear resonances, each being a chain of q_m stable domains around the period–q_m trajectory surrounded by a relatively thick chaotic layer (see, e.g., [9–11] for details). The chain goes along the chaos border, and its transverse size ρ_m and area A_m are given by the estimates:

$$A_m \sim \frac{A(K)}{q_m^2} \propto \rho_m \propto S_m \tag{7}$$

Here $A(K)$ is the full islet area (5), and $S_m = S(\nu_m)$ stands for the amplitude of transverse oscillation (bending) of the chaos border at frequency $\nu_m = q_m|r_b - r_m| \sim 1/q_m$. Whence,

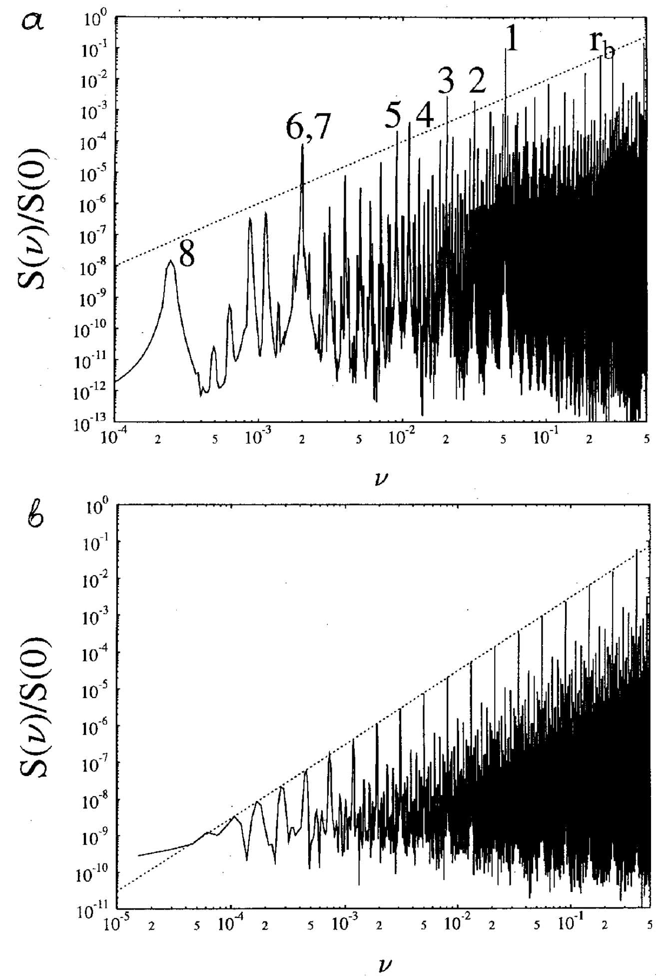

Figure 2: Two examples of motion spectrum on chaos border: ν ($mod\, 1$) is frequency, and $S(\nu)/S(0)$ stands for the relative Fourier amplitude; the total motion time $T = 65536$ iterations; dotted line is theory (8). (a) Statistical scaling (chaotic renormgroup) on robust chaos–order border: $n = 1$; $r_b = 0.23713...$ (peak r_b) which is slightly different from asymptotic value (6); serial numbers m of basic resonances are shown. (b) Regular scaling (fixed point of renormgroup) on nonrobust chaos–chaos border for special $r_b = (3 - \sqrt{5})/2 = [2, 1, 1, 1, ...]$ and $K = K_{cr}$.

the global shape of the spectrum

$$S_m \sim \nu_m^2 \tag{8}$$

shown by the dotted line in Fig.2. In case of the renormchaos this simple relation represents, of course, an average behavior only, superimposed on strong fluctuations which are generally characteristic for the critical phenomena.

The diffusion rate is determined by the statistic of sticking times t_m in the corresponding scale m. In average over time or the initial conditions (ergodicity), and assuming statistical independence of stickings

$$< (\Delta p)^2 > \approx \sum_m (\Delta p)_m^2 \approx K^2 \sum_m t_m^2 \, N_m + \frac{K^2}{2} C_0(K) \, t \tag{9}$$

Here $N_m(t)$ is the number of entries into scale m for the total motion time t, and the latter term describes the normal diffusion (with additional coefficient $C_0(K) \sim 1$ which accounts for the short–time correlations). The normal diffusion occupies the most of the time owing to a small size of regular domains. In turn, the number of entries

$$N_m = t \cdot P_m \, , \qquad P_m \sim \frac{A}{t_m^{c_P}} \tag{10}$$

where $P_m = P(t_m)$ stands for the Poincare recurrence statistic that is the distribution of the delays during the reflection (scattering) from the chaos border which is characterized by the critical exponent c_P. From the motion ergodicity

$$\frac{t_m N_m}{t} = t_m P_m = A_m \sim \frac{A}{q_m^2} \sim \frac{A}{t_m^{c_A}} \tag{11}$$

Function $A_m = A(t_m)$ plays a role of the sticking correlation, and from the latter estimate (scaling) $c_P = c_A + 1$. Whence, the asymptotic average diffusion rate

$$D(t) \equiv \frac{< (\Delta p)^2 >}{t} \rightarrow K^2 \sum_m t_m \, A_m \sim AK^2 \, t_{max}^{1-c_A} \sim AK^2 \cdot (At)^{c_D} \tag{12}$$

where the critical diffusion exponent

$$c_D = \frac{1 - c_A}{1 + c_A} \tag{13}$$

and the maximal sticking time t_{max} is determined from the condition

$$N_m(t_{max}) \sim \frac{A \, t}{t_{max}^{c_P}} \sim 1 \, , \qquad \left(\frac{t_{max}}{t} \right)^{c_P} \sim \frac{A}{t^{c_A}} \ll 1 \tag{14}$$

The sum in (12) approximately amounts to the biggest term with $t_m = t_{max}$ because all the quantities of the critical structure depend exponentially on scale number m (geometrical progression). Of course, this holds true under condition $c_D < 1$ only. Otherwise, the diffusion rate does not depend on time that is the diffusion becomes normal.

The theory of critical exponents at the chaos border proved to be rather nontrivial. It requires estimating the dependence $t_m(q_m)$. At the first glance, it seems natural to assume $t_m \sim q_m$ that is the sticking (exit) time is of the order of characteristic time for a given scale [12]. However, it is immediately clear from Eq.(7) that in such a case $c_A = 2$, and $c_P = 3$ which is completely incompatible with the well measured exponent

$c_P \approx 1.5$ [9,11,13,14]. Moreover, besides this quantitative disagreement a qualitatively different result would follow for the diffusion [11], namely, the latter were normal in spite of sticking.

This qualitative effect is especially important for the evaluation of farther development of the approach [12] in papers [15]. Such a theory is essentially based on the analysis of internal chaos borders which also have a hierarchical structure ("resonances around resonances", see also [16,17]). The value of the critical exponent, most close to the empirical one, which has been achieved in this approach ($c_P \approx 2$ [15]) was still too large to allow anomalous diffusion. On the other hand, it was shown in [9] that for $c_P < 2$ the contribution from internal chaos borders does not influence the critical exponents at all. However, it should be mentioned that time scale $t_m \sim q_m$ has, nevertheless, some physical meaning, and not only well known dynamical one (the period of basic resonance) but also statistical, related to the rate of the local diffusion transverse to the chaos border: $D_l \sim q_m^{-4}/q_m = q_m^{-5}$ [18]. Such a diffusion has been observed recently, indeed [16]. Yet, it is bounded, and only leads to the local statistical equilibrium without any transition into neighboring scales.

For resolution of this apparent contradiction the following hypothesis has been put forward in [11]: in the exact criticality all $t_m = \infty$ that is all the scales of the critical structure are dynamically isolated, or separated by their own chaos borders, the invariant curves. Notice that in a hierarchical critical structure the latter form an everywhere dense set. According to this hypothesis the finite t_m are explained by a deviation of the local order parameter, in vicinity of chaos border, from the critical value on the proper border. In this way, the structure becomes subcritical on the one side of the border, thus providing regular motion for the host of initial conditions, and supercritical on the other side, which results in a finite sticking time. Assuming linear dependence of the local order parameter on the distance from the border the following estimate was derived:

$$t_m \sim q_m^{c_t}\,, \qquad c_A = \frac{2}{c_t} \tag{15}$$

Depending on the details of the critical structure $c_t = 7$ [11] or $c_t = 4$ [9], and hence $c_A = 2/7$ or $c_A = 1/2$, respectively. The latter value seems to me more accurate (see [9] for discussion).

In both cases $c_A < 1$ that is the diffusion is anomalous (enhanced, $c_D > 0$) anyway. Already the first numerical experiments [11] did confirm the existence of anomalous diffusion on the chaos border, and hence did refute both the initial guess $c_t = 1$ as well as its further development [15]. Later, such a diffusion was studied in many papers (see, e.g., [19,20,17]).

A distinctive feature of the model in the present paper is a very small size of the stable domain and, hence, of the chaos border which, nevertheless, does crucially change the statistical properties for a sufficiently long time. This emphasizes the importance as well as universality of the robust critical structure.

It should be mentioned that the anomalous diffusion in a broad sense, both enhanced one ($c_D > 0$) as well as suppressed ($c_D < 0$), was studied even much earlier (see, e.g., [21]). Here, we are interested in a particular diffusion caused by the specific critical structure at the chaos border.

4 Numerics

In the model under consideration the main difficulty of empirical studies of anomalous diffusion comes from big fluctuations. These are related to the diffusion peculiarity when the essential contribution comes from a single (for a given time) sticking with $t_m = t_{max}$. For the same reason the fluctuations are growing with time (Fig.3). Suppression of those was done by a double averaging: first, by standard averaging $D(t)$ over $M = 40$

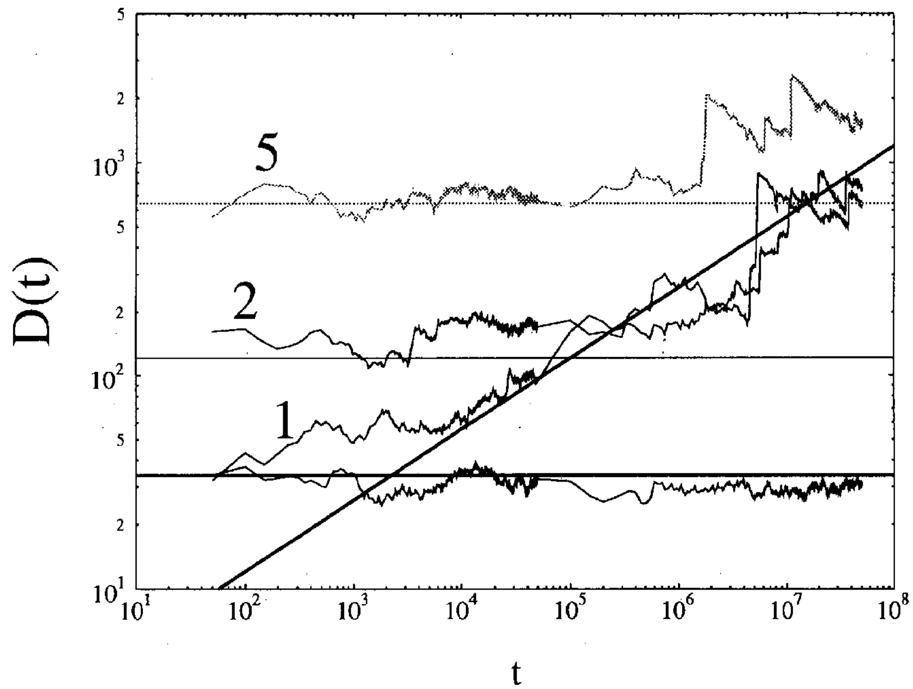

Figure 3: Anomalous diffusion for model (1) in microtron regime (3): broken lines show numerical data for $n = 1,\ 2,\ 5$ as indicated, and for additional run with $K = 2\pi$ (see text); horizontal straight lines represent (constant) rate of the normal diffusion; oblique line is dependence (19) with theoretical $c_D = 1/3$ and empirical $b = 11$ (for $n = 1,\ M = 40$).

independent trajectories, and then by taking the additional average of $c_D(t)$ over four also independent groups of trajectories.

The main results of numerical experiments are presented in Fig.3 for $n = 1$, $K_1 = 6.5938...$, $D_0 = C_0K^2/2 \approx 39$; $n = 2$, $K_2 = 12.72...$, $D_0 \approx 121$ and $n = 5$, $K_5 = 31.47...$, $D_0 \approx 644$. An example of normal diffusion with $K_0 = 2\pi$ is also shown. In the latter case the stability domain gets complitely destroyed (3), and the diffusion becomes normal over the whole run $t \le 5 \cdot 10^7$ iterations in spite of only a minor change in K ($K_1/K_0 - 1 \approx 0.05$).

Thus, there is no more doubt as to the existence of anomalous diffusion due to the chaos border. However, evaluation of the critical exponent c_D is a more difficult task because of big fluctuations discussed above. "Levy's flights" caused by trajectory sticking at border are clearly seen in Fig.3. Remarkably, the flights are rather asymmetric in the slope, the peculiarity whose mechanism is not yet completely clear. The asymptotic regime of anomalous diffusion is reached in some time, the longer the smaller is the islet size. Asymptotic dependence $D(t)$ ($t > t_a$) was fitted in log–log scale to the linear expression (see (12)):

$$\ln D(t) \; = \; c_D \cdot \ln t \, + \, \ln B \tag{16}$$

The results are as follows

$$\begin{array}{llll} n=1, & c_D = 0.29 - 0.37, & B = 4.4 - 1.3, & t_a = 5\cdot 10^3 - 10^5 \\ n=2, & c_D = 0.34 - 0.39, & B = 2.2 - 0.9, & t_a = 5\cdot 10^4 - 5\cdot 10^5 \end{array} \tag{17}$$

The minimal t_a value corresponds to the beginning of asymptotic regime while the maximal one is limited by big jumps in $D(t)$ (see Fig.3). Even though the anomalous diffusion with its characteristic jumps is clearly evident up to $n = 5$ the allowed computer time was insufficient, in the latter case, to reach the asymptotic regime, and to measure c_D with a reasonable accuracy. For $n = 1, 2$ evaluated c_D values (17) are in a good agreement with each other, and with theoretical $c_D = 1/3$ which is shown in Fig.3. According to (12) the second fitting parameter B can be expressed in the form:

$$B_n = b\cdot A_n K_n^2 \cdot (A_n M)^{c_D} \approx \frac{0.07\, bM^{1/3}}{n^{2/3}}, \quad t^{c_A} \gtrsim f\cdot MA_n \tag{18}$$

where A_n now denotes the total area of the two islets for a given n, and b, f are some factors. The dependence on the number of trajectories M is again related to a peculiarity of the anomalous diffusion, namely, the sticking of even a single trajectory is already sufficient if the latter condition in (18) is fulfilled. Otherwise, $t_m \sim t$, and $c_D \approx c_A \approx 1/2$. Notice that t, not Mt, enters the inequality since the trajectories are independent. The latter expression for B_n in (18) is obtained taking account of (5), and of the accepted value $c_D = 1/3$. From data (17) and for $M = 40$ we arrive at: $b \approx 11$ ($n = 1,\ B \approx 2.8$) and $b \approx 10$ ($n = 2,\ B \approx 1.5$). Finally, the anomalous diffusion in question is described by approximate relation:

$$D_a(t) \approx 0.76\left(\frac{Mt}{n^2}\right)^{1/3} \tag{19}$$

which is shown in Fig.3 for $n = 1$ by oblique straight line.

Now, let us compare the results obtained with previous data. First of all, the theoretical value $c_D = 1/2$, given in [22], is somewhat larger which is explained by a simplified assumption $t_{max} \sim t$ (cf. (14)) used in [22] following [11,9]. Such an assumption is only true in case of violation of inequality (18), particularly, for very big M and small t (see below).

In numerical experiments the anomalous diffusion on the chaos border was apparently first observed in [23] for the same model with $n = 1$. However, the diffusion rate was given for the maximal run time $t = 10^8$ only. Strangely, the authors [23] were unable to see anomalous diffusion for $n = 2$, even though its rate is about the same at $t = 10^8$, and still considerably exceeds the normal rate: $D/D_0 \approx 7.3$ (see (19), $M = 64$, and Fig.3). The ratio $D/D_0 \approx 27$ found in [23] for $n = 1$ is in a reasonable agreement with mean $D/D_0 \approx 36$ (19) on the background of big fluctuations, also mentioned in [23].

Further studies of anomalous diffusion for the same model but with somewhat different value of $K = 6.9115$, and on a very short time interval $t \leq 2000$ were presented in [20]. The critical exponent value $c_D = 1/3$ was accepted but its accuracy remained unclear, essentially because of a very short t. At the maximal run time $D \approx 160$ while theoretical value (19) is $D \approx 60$ (taking into account that the islet area is 5 times smaller, see (18)). Apparently, the difference is not so much due to fluctuations as because of a different form of the islet and, hence, of another rotation number on the chaos border. In [20] a huge number of trajectories $M = 10^5$ was used for calculation of the diffusion distribution function. Interestingly, the results implicitly confirm a strange, at first glance, dependence of the mean diffusion rate on M (19). Without such a factor the rate would drop by almost 50 times! A more careful analysis of data in [20], checked by additional numerical experiments, reveals a transition at $t \sim 100$ from $c_D \approx 0.5$ to $c_D \approx 0.3$, apparently related to violation of inequality (18). This allows for estimating factor $f \sim 0.05$.

Similar results were obtained for another model (in continuous time) [17] with $M = 3600$ and maximal $t \sim 10^5$ (in comparable units). Particularly, a close value of the critical exponent $c_D \approx 0.38 \approx 1/3$ has been found in spite of completely different global structure of the motion. It gives one more confirmation for universality of the critical structure at the chaos border. Also, a decrease in c_D with time around $t \sim 10^4$ is even more clear here (see Fig.7 in [20]), and it gives roughly the same value for f assuming $A \sim 1$.

Thus, the existence of anomalous diffusion due to the critical structure at chaos border can be taken as reliably proven. However, the very existence of such a diffusion ($c_D > 0$) as well as approximate evaluation of the critical exponents is only possible, so far, in the resonance theory of chaos critical structure with the additional hypothesis on supercriticality of the whole structure but the border [11,9]. This important hypothesis was further confirmed, at least qualitatively, on a different model with the chaos–chaos border.

5 Statistical properties of motion with chaos–chaos border

Unlike the more known and robust chaos–order border, which persists within a relatively broad range of the order parameter (K in the model under consideration), the chaos–chaos border is not robust, i.e. it gets destroyed by any deviation from the critical value $K = K_{cr} = 0.9716...$ [10]. A phase portrait for model (1) with $K = K_{cr}$ is shown in Fig.4. Two critical invariant curves, marked by arrows, are absolute barriers for the motion, yet chaotic trajectories come arbitrarily close to each of them and, moreover,

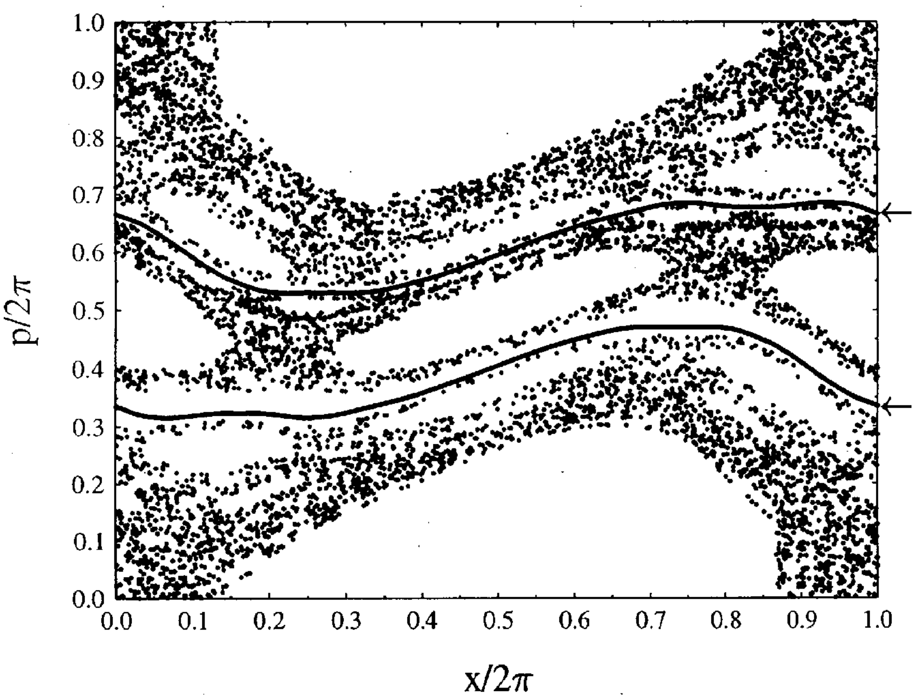

Figure 4: Phase portrait for model (1) at critical $K = K_{cr}$: arrows indicate two chaos–chaos borders separating chaotic components, the motion in each of them being represented by a single trajectory on $t = 10^7$ in steps $\Delta t = 2000$ and 5000.

from both sides. Now, the local order parameter is supercritical also on both sides. Hence, it is approaching the critical value on the border at least quadratically, and the whole supercriticality sharply drops, as compared to the case of chaos–order border, while the sticking time considerably increases. In turn, this results in a decrease of critical correlation exponent $c_A \to 0$, and in increase of the diffusion exponent up to its limiting value: $c_D \to 1$. The first confirmation of such a structure has been obtained already in [11] from the measurement of the Poincare recurrence statistic (10), namely, the critical exponent $c_P = 1 + c_A \approx 0.975 \pm 0.013$ was found to be very close to its limiting value, indeed.

The results on the anomalous diffusion in this case are presented in Fig.5. The mean

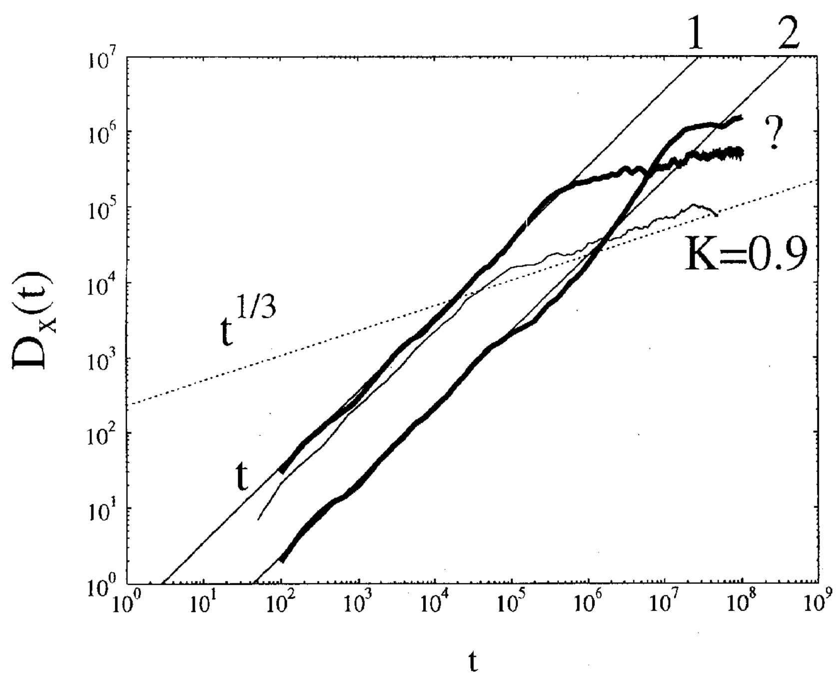

Figure 5: Anomalous diffusion on chaos–chaos border in model (1) with critical K_{cr}, averaging over $M = 100$ independent trajectories. Thick curves correspond to the limiting–rate diffusion ($D(t) \propto t$, steep straight lines) for integer (1) and half–integer (2) resonances. Diffusion for subcritical $K = 0.9$ (thin line) is also shown for comparison. Dotted line corresponds to $c_D = 1/3$ for diffusion on chaos–order border.

diffusion rate in x was defined as

$$D_x(t) = \frac{< (\Delta x)^2 >}{t} \approx B\, t^{c_D}, \qquad \Delta x = \sum_t (p(t) - p_r) \tag{20}$$

where the resonant momentum value $p_r = p_1 = 0$ for diffusion on the integer resonance (under lower border in Fig.4), and $p_r = p_2 = \pi$ for diffusion on the half–integer resonance (between the two borders).

In this case the map (1) models another physical system: particle motion in a multi–wave field in Cartesian variables x, p. Particularly, x–motion is now unbounded. Such a model was studied in [17] in case of two waves only. This is the simplest limit from the physical viewpoint but a much more complicated one for numerical experiments and the theoretical analysis.

Numerical data in Fig.5 show that diffusion exponent $c_D \approx 1$ considerably increases in comparison to the model with chaos–order border (Fig.3), and it becomes close to the limiting value, indeed. If the trajectory stuck at the chaos border for the whole motion time t the factor in (20) would be $B_i \approx (2\pi r_i)^2$ where $r_1 = (3-\sqrt{5})/2 \approx 0.382$ is the rotation number on the lower chaos border, and $r_2 = 0.5 - r_1$. Whence, $B_1 \approx 5.8$ and $B_2 \approx 0.55$. Actually, from the data in Fig.5, $B_1 \approx 0.35$, $B_2 \approx 0.023$ that is approximately 20 times less. Apparently, it is mainly explained by a relatively small size of the proper critical structure ($A \sim 0.1$, cf. parameter f of same order in inequality (18)). On the other hand, the latter area is still much larger than in the microtron model which apparently results in considerably lower fluctuations (cf. Figs.3 and 5). In any case, small B values indicate that x–motion is diffusive, not regular one, even though $x \propto t$ for both mechanisms. This is also confirmed by the direct spying of trajectories. Particularly, such a "quasiregular" motion goes in both directions ($x \to \pm\infty$)! If $c_P \approx 1$ and $c_A \approx 0$ the inequality (18) is violated already for $M \gtrsim 1/f \sim 10$, and the diffusion rate does not depend on M. Notice that in both cases $c_D \approx (1-c_A)/(1+c_A) \approx 1 - c_A \approx 1$.

Thus, the limiting diffusion on chaos–chaos border does confirm the supercriticality hypothesis which seems to be very important for further development of the theory of critical phenomena in dynamical systems. However, the nature of stable (independent on initial conditions) anomaly in the anomalous diffusion for large $t \gtrsim 10^6$ remains as yet completely unclear. The anomaly is especially evident for case 1 in Fig.5. A trace of such anomaly was noticed already in [11] as a more rapid decrease in Poincare recurrence distribution function $P(t)$ for $t \gtrsim 10^5$. The latter anomaly turns out to be stable as well, and seems unrelated to a poor statistic as was conjectured in [11].

The anomaly in Fig.5 looks as if the critical parameter value K_{cr} were still subcritical (cf. case $K = 0.9$) or the sticking switches, for some reason, from the main chaos–chaos border to one of the internal chaos–order borders. However, no anomaly is seen in the motion spectrum on the chaos border (Fig.2b). Unfortunately, it cannot be excluded at the moment that the anomaly is simply a result of computation (rounding–off) errors whose impact happens to be very surprising sometimes. In any case, the single–precision computation (of course, all the computation was normally done in double precision) does not increase the anomaly, as one might expect, but to the contrary it apparently removes it! This doesn't mean, of course, that the single–precision results are more correct. The whole problem certainly requires special investigation.

6 Conclusion

A large series of numerical experiments with simple dynamical system (1) was carried out aiming to the study of the critical structure on the chaos–order border (Fig.1) by observing a peculiar anomalous diffusion caused by such a structure. Particularly, the value of critical exponent $c_D \approx 1/3$ in anomalous diffusion law (12) has been measured to a reasonably good accuracy, and found to be close to one predicted by the theory of critical phenomena [11,9]. The studies were done in a specific (microtron) regime (3) of model (1) when the chaos border is of a very small size, yet it does completely determine the statistical properties of the whole motion chaotic component for a sufficiently large time. This emphasizes importance of the critical phenomena in dynamics, especially in view of robustness (structural stability) of the chaos–order border.

One of the main goals of investigation was confirmation of the basic hypothesis in the theory concerning supercriticality of the local order parameter on the chaotic side of the border. Evaluated empirical c_D value and the related critical exponent of correlation on the chaos border $c_A \approx 1/2$ (13) do well confirm this hypothesis. In view of importance of the latter for the whole theory an additional verification has been carried out with the different (critical) value of order parameter $K = K_{cr}$ for which the (nonrobust) chaos–chaos border arises with a qualitatively different structure (Figs.1,4,2). Sharp increase in $c_D \to 1$ and drop in $c_A \to 0$ has been confirmed (Fig.5). At the same time, a stable anomaly was found (noticed already in [11]) which is currently under study. Interestingly, in the case of chaos–chaos border the rate of (homogeneous) diffusion reaches the upper bound: $|\Delta x| \propto t$ that is the motion looks like a free one but goes on with a smaller speed and in both directions!

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