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Abstract

The helicity amplitudes of the high-energy photon splitting in the external Coulomb field are obtained exactly in the parameter $Z\alpha$. The cases of screened and unscreened potentials are investigated. The consideration is based on the quasiclassical approach, valid for small angles between all photon momenta. New representation of the quasiclassical electron Green function is exploited. General expressions obtained are analyzed in detail for the case of large transverse momenta of both final photons compared to the electron mass.

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1 Introduction

It is well known, that the virtual electron-positron pair creation in an external Coulomb field gives rise to such nonlinear QED phenomena as Delbrück scattering(coherent photon scattering [1]) and the splitting of one photon into two. At present the process of Delbrück scattering has been studied in detail both theoretically and experimentally (see recent review [2]). At high photon energies $\omega \gg m$ (m is the electron mass, $\hbar = c = 1$) the scattering amplitude has been found exactly in the parameter $Z\alpha$ ($Z|e|$ is the nucleus charge, $\alpha = e^2/4\pi = 1/137$ is the fine-structure constant, e is the electron charge). The approaches used essentially depended on the momentum transfer $\Delta = |\mathbf{k}_2 - \mathbf{k}_1|$ ($\mathbf{k}_1, \mathbf{k}_2$ being the momenta of the initial and final photons, respectively). The main contribution to the total cross section of Delbrück scattering comes from small momentum transfers $\Delta \ll \omega$ (the scattering angle $\theta \sim \Delta/\omega \ll 1$). In this case the amplitudes have been found in [3, 4, 5] by summing in a definite approximation the diagrams of perturbation theory with respect to the interaction with the Coulomb field, and also in [6, 7] within the quasiclassical approach. It turned out that at $\omega \gg m$ and $Z\alpha \sim 1$ the exact in $Z\alpha$ result significantly differs from that obtained in the lowest order of the perturbation theory.

The applicability of the quasiclassical approximation is based on the fact that, according to the uncertainty relation, the characteristic impact parameter is $\rho \sim 1/\Delta$ and the corresponding angular momentum is $l \sim \omega\rho \sim \omega/\Delta \gg 1$ at small scattering angles. This fact was used in Refs. [6, 7], where the quasiclassical Green function has been derived from the integral representation for the Green function of the Dirac equation in the Coulomb field [8]. In Refs. [9, 10] the quasiclassical Green function of an electron for an arbitrary decreasing spherically symmetric potential has been obtained, which allowed one to calculate the Delbrück scattering amplitudes in a screened Coulomb potential.

So far the process of photon splitting has not been observed, although some events in the experiment performed at DESY [11] were erroneously interpreted as photon splitting. As it was shown in [12] these events were due to the electron-positron pair production accompanied by hard-photon bremsstrahlung. Some possibilities to observe photon splitting have been discussed in [13]. Photon splitting has been investigated theoretically in [14, 15, 16, 17, 18] in the lowest order of perturbation theory with respect to the parameter $Z\alpha$. The expressions obtained in [14, 15] are rather cumbersome and it is difficult to use them for numerical calculations. Nevertheless, some calculations based on the results of Ref. [14, 15] have been carried out in [17, 18]. Using the Weizsäcker-Williams method providing the logarithmic accuracy the amplitudes of the process have been derived in an essentially simpler form in Ref. [16]. The comparison of the exact cross section [17] with the approximate result [16] has shown that at high photon energy the accuracy is better than 20%. The magnitude of the Coulomb corrections to the lowest order amplitude of photon splitting has been unknown up to now. At the present time the experiment dedicated to the observation of high-energy photon splitting ($\omega \gg m$) in a strong Coulomb field is held in the Budker Institute of Nuclear Physics (Novosibirsk). Therefore, the theoretical investigation of the problem is of great interest.

In the present paper the high-energy photon-splitting amplitude is calculated exactly in $Z\alpha$ for small

angles f_2 and f_3 between the momenta \mathbf{k}_2 , \mathbf{k}_3 of the final photons and the momentum \mathbf{k}_1 of the initial one. It is the region of small angles that gives the main contribution to the total cross section of the process. Besides, small angles and high energies of photons allow one to use the quasiclassical approach developed in [6, 7, 9, 10] at the consideration of Delbrück scattering. We consider the case of a pure Coulomb potential as well as the influence of screening. The initial representation for the splitting amplitude is rather cumbersome and contains a thirteen-fold integral. The quasiclassical approach gives the transparent picture of the phenomenon and allows one to determine the region of integration which gives the main contribution to the amplitude. Without that it seems impossible to calculate the amplitude.

Our paper is organized as follows. In Sec. II we perform some transformations of the exact amplitude. These transformations essentially simplify further calculations. Section III contains the discussion of the process kinematics. In Sec. IV the small-angle approximation for the quasiclassical Green function is derived. In Sec. V this Green function is applied to the calculation of the photon-splitting amplitude. In Sec. VI we consider the case of large transverse momenta of both final photons ($\omega_2 f_2 \gg m$, $\omega_3 f_3 \gg m$). The limiting cases of small momentum transfers and $Z\alpha \ll 1$ for this amplitude are studied in Sec. VII and Sec. VIII, respectively. In the last Section the cross sections obtained in the Born approximation are presented and compared to those obtained within the Weizsäcker-Williams method.

2 Transformation of the amplitude

According to the Feynman rules the photon-splitting amplitude in the Furry representation can be written as:

$$M = ie^3 \int d^4x \text{Tr} \langle x | \hat{e}_1 e^{-ik_1 x} \mathcal{G} \hat{e}_2^* e^{ik_2 x} \mathcal{G} \hat{e}_3^* e^{ik_3 x} \mathcal{G} | x \rangle + \{k_2^\mu \leftrightarrow k_3^\mu, e_2^\mu \leftrightarrow e_3^\mu\}. \quad (1)$$

Here e_1^μ and $e_{2,3}^\mu$ are the polarization vectors of the initial and final photons, respectively, $\hat{e} = e^\mu \gamma_\mu = -\mathbf{e}\boldsymbol{\gamma}$, γ^μ being the Dirac matrices, $\mathcal{G} = 1/(\hat{\mathcal{P}} - m + i0)$, and $\mathcal{P}_\mu = i\partial_\mu + g_{\mu 0}(Z\alpha/r)$. The matrix element of the operator \mathcal{G} is the Green function of the Dirac equation in the Coulomb field: $G(x, x') = \langle x | \mathcal{G} | x' \rangle$.

It is convenient to rewrite the expression (1) in the form, containing only the Green functions of the 'squared' Dirac equation:

$$D(x, x') = \langle x | \mathcal{D} | x' \rangle = \langle x | 1/(\hat{\mathcal{P}}^2 - m^2 + i0) | x' \rangle.$$

For this purpose we represent the left operator \mathcal{G} in (1) in the form $\mathcal{G} = \mathcal{D}(\hat{\mathcal{P}} + m)$ and use the commutative relation

$$(\hat{\mathcal{P}} + m)\hat{e} e^{ikx} = e^{ikx} [-\hat{e}\mathcal{G}^{-1} + \hat{e}\hat{k} - 2\mathbf{e}\mathbf{p}].$$

One can obtain another expression by similar transformation of the right operator \mathcal{G} in (1). Taking a half-sum of these two expressions and using the identity $\int dx \text{Tr} \langle x | A_1 A_2 | x \rangle = \int dx \text{Tr} \langle x | A_2 A_1 | x \rangle$, valid for arbitrary operators A_1 and A_2 , we get:

$$M = ie^3 \int d^4x \text{Tr} \left\{ \left[\begin{aligned} & \mathbf{e}_1 \mathbf{e}_2^* e^{i(k_2 - k_1)x} \langle x | \mathcal{D} e^{ik_3 x} (\hat{e}_3^* \hat{k}_3 - 2\mathbf{e}_3^* \mathbf{p}) \mathcal{D} | x \rangle + \\ & \mathbf{e}_1 \mathbf{e}_3^* e^{i(k_3 - k_1)x} \langle x | \mathcal{D} e^{ik_2 x} (\hat{e}_2^* \hat{k}_2 - 2\mathbf{e}_2^* \mathbf{p}) \mathcal{D} | x \rangle + \\ & \mathbf{e}_2^* \mathbf{e}_3^* e^{-i(k_2 + k_3)x} \langle x | \mathcal{D} e^{-ik_1 x} (-\hat{e}_1 \hat{k}_1 - 2\mathbf{e}_1 \mathbf{p}) \mathcal{D} | x \rangle \end{aligned} \right] + \right. \\ \left. \frac{1}{2} \left[\langle x | e^{-ik_1 x} (-\hat{e}_1 \hat{k}_1 - 2\mathbf{e}_1 \mathbf{p}) \mathcal{D} e^{ik_2 x} (\hat{e}_2^* \hat{k}_2 - 2\mathbf{e}_2^* \mathbf{p}) \mathcal{D} e^{ik_3 x} (\hat{e}_3^* \hat{k}_3 - 2\mathbf{e}_3^* \mathbf{p}) \mathcal{D} | x \rangle + (2 \leftrightarrow 3) \right] \right\}. \quad (2)$$

The formula (2) represents the amplitude as a sum of the terms containing either two or three Green functions: $M = M^{(2)} + M^{(3)}$. The amplitude in the form of eq. (1) contains terms of different order of

magnitude and a strong compensation occurs. After taking the trace in eq. (2) the expression obtained contains only the terms of the required order what is the advantage of this formula.

Passing from the time-dependent Green functions to the energy-dependent ones, taking the integral over time in eq (2) and omitting the conventional factor $2\pi\delta(\omega_1 - \omega_2 - \omega_3)$, we get for the contribution to M containing three Green functions

$$M^{(3)} = \frac{i}{2}e^3 \int \frac{d\varepsilon}{2\pi} \int d\mathbf{r}_1 d\mathbf{r}_2 d\mathbf{r}_3 \exp[i(\mathbf{k}_1\mathbf{r}_1 - \mathbf{k}_2\mathbf{r}_2 - \mathbf{k}_3\mathbf{r}_3)] \times \quad (3)$$

$$\text{Tr} \left\{ [(-\hat{e}_1 \hat{k}_1 - 2\mathbf{e}_1 \mathbf{p})D(\mathbf{r}_1, \mathbf{r}_2 | \varepsilon - \omega_2)] [(\hat{e}_2^* \hat{k}_2 - 2\mathbf{e}_2^* \mathbf{p})D(\mathbf{r}_2, \mathbf{r}_3 | \varepsilon)] \times \right.$$

$$\left. [(\hat{e}_3^* \hat{k}_3 - 2\mathbf{e}_3^* \mathbf{p})D(\mathbf{r}_3, \mathbf{r}_1 | \varepsilon + \omega_3)] \right\} + (k_2^\mu \leftrightarrow k_3^\mu, \mathbf{e}_2 \leftrightarrow \mathbf{e}_3).$$

Here $\mathbf{p} = -i\nabla$ differentiates the Green function D of the 'squared' Dirac equation with respect to its first argument.

Below the Green function $D(\mathbf{r}_1, \mathbf{r}_2 | \varepsilon)$ will be called the 'electron Green function' for $\varepsilon > 0$ and the 'positron Green function' for $\varepsilon < 0$. Let the initial photon propagates along z axis. Then according to the quasiclassical approach developed in [6, 7, 9, 10] the main contribution to the amplitude at high energy arises from the region of integration over the variables z_i in which $z' < z$ for the electron Green function $D(\mathbf{r}, \mathbf{r}' | \varepsilon)$ ($\varepsilon > 0$) and $z' > z$ for the positron Green function ($\varepsilon < 0$). In terms of noncovariant perturbation theory this range of variables corresponds to the contribution of intermediate states for which the difference between their energy E_n and the energy of the initial state $E_0 = \omega_1$ is small compared to E_0 . Outside this region at least for one of the intermediate states $|E_n - E_0| \sim E_0$ and therefore the corresponding contribution is suppressed. Besides, there exist another restriction for the region giving the main contribution to the amplitude. This restriction follows from the explicit form of the quasiclassical Green function. It reads

$$z_1 < z_2, z_3 \quad , \quad z_1 < 0 \quad , \quad \max(z_2, z_3) > 0.$$

All these conditions allow one to depict the main contribution to the amplitude $M^{(3)}$ in the form of diagrams, shown in Fig. 1. The explicit form of vertices is obvious from eq. (3). The electron Green

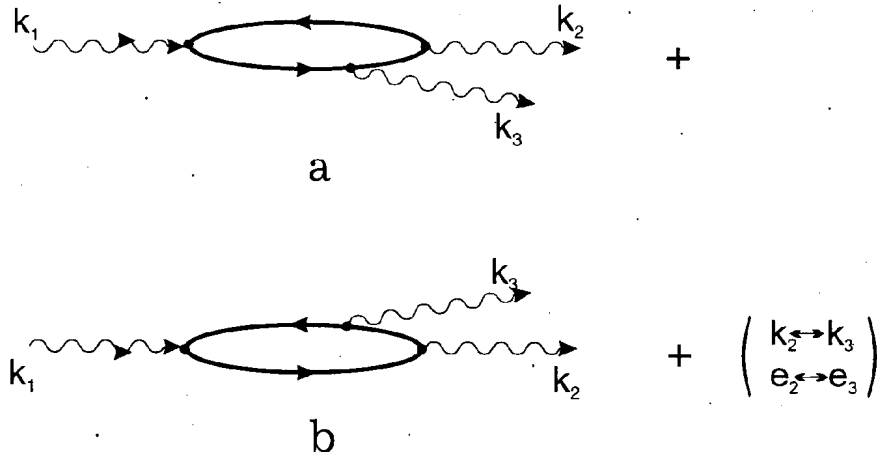


Figure 1: Diagrams of the perturbation theory corresponding to the amplitude $M^{(3)}$, eq. (3).

functions are marked with left-to-right arrows, and positron ones with right-to-left arrows. The arrangement of the diagram vertices is space ordered. With the use of these diagrams one can easily determine the limits of integration over the energy and coordinates. The diagrams (a) and (b) correspond to the following picture: the photon with momentum \mathbf{k}_1 produces at the point \mathbf{r}_1 a pair of virtual particles which is transformed in the point \mathbf{r}_2 into a photon with momentum \mathbf{k}_2 . Between these two events the electron (a) or the positron (b) emits a photon with the momentum \mathbf{k}_3 at the point \mathbf{r}_3 .

Analogously, the expression for the term $M^{(2)}$ containing two Green functions reads

$$M^{(2)} = ie^3 \int \frac{d\varepsilon}{2\pi} \int d\mathbf{r}_1 d\mathbf{r}_2 \text{Tr} \left\{ \exp[i(\mathbf{k}_1 \mathbf{r}_1 - \mathbf{k}_2 \mathbf{r}_2 - \mathbf{k}_3 \mathbf{r}_2)] \mathbf{e}_2^* \mathbf{e}_3^* \times \right. \quad (4)$$

$$\left. [(-\hat{\varepsilon}_1 \hat{\mathbf{k}}_1 - 2\mathbf{e}_1 \mathbf{p}) D(\mathbf{r}_1, \mathbf{r}_2 | \varepsilon - \omega_1)] D(\mathbf{r}_2, \mathbf{r}_1 | \varepsilon) + \right.$$

$$\left. \left[\exp[i(\mathbf{k}_1 \mathbf{r}_1 - \mathbf{k}_2 \mathbf{r}_2 - \mathbf{k}_3 \mathbf{r}_1)] \mathbf{e}_1 \mathbf{e}_3^* D(\mathbf{r}_1, \mathbf{r}_2 | \varepsilon - \omega_2) [(\hat{\varepsilon}_2^* \hat{\mathbf{k}}_2 - 2\mathbf{e}_2^* \mathbf{p}) D(\mathbf{r}_2, \mathbf{r}_1 | \varepsilon)] + (k_2^\mu \leftrightarrow k_3^\mu, \mathbf{e}_2 \leftrightarrow \mathbf{e}_3) \right] \right\}.$$

The diagrams, corresponding to the representation (4), are shown in Fig. 2.

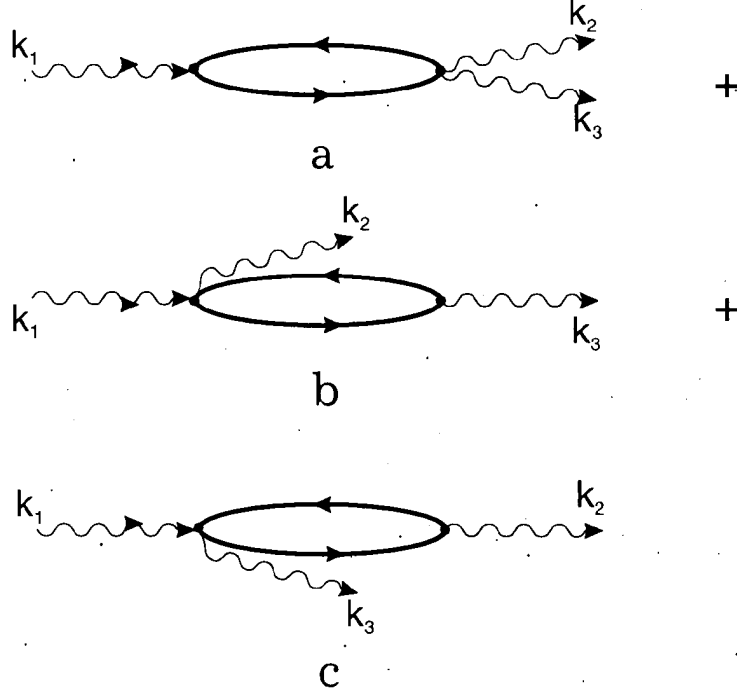


Figure 2: Diagrams of the perturbation theory corresponding to the amplitude $M^{(2)}$, eq. (4).

3 Kinematics of the process

According to the uncertainty relation the lifetime of the virtual electron-positron pair is $\tau \sim |\mathbf{r}_2 - \mathbf{r}_1| \sim \omega_1 / (m^2 + \tilde{\Delta}^2)$, where $\tilde{\Delta} = \max(|\mathbf{k}_{2\perp}|, |\mathbf{k}_{3\perp}|) \ll \omega_1$, $\mathbf{k}_{2\perp}$ and $\mathbf{k}_{3\perp}$ being the transverse components of the final photon momenta. The characteristic transverse distance between the virtual particles can be estimated as $(m^2 + \tilde{\Delta}^2)^{-1/2}$, which is much smaller than the length of the electron-positron loop. The characteristic impact parameter is $\rho \sim 1/\Delta$, where $\Delta = \mathbf{k}_2 + \mathbf{k}_3 - \mathbf{k}_1$ is the momentum transfer. At small $\mathbf{k}_{2\perp}$ and $\mathbf{k}_{3\perp}$ ($f_{2,3} \ll 1$) we have

$$\Delta^2 = (\mathbf{k}_{2\perp} + \mathbf{k}_{3\perp})^2 + \frac{1}{4} \left(\frac{\mathbf{k}_{2\perp}^2}{\omega_2} + \frac{\mathbf{k}_{3\perp}^2}{\omega_3} \right)^2. \quad (5)$$

The characteristic angular momentum is $l \sim \omega/\Delta \gg 1$, and the quasiclassical approximation can be applied.

Let us discuss a screened Coulomb potential. In the Thomas-Fermi model the screening radius is $r_c \sim (m\alpha)^{-1} Z^{-1/3}$. If $R \ll 1/\Delta \ll r_c$ (R is the nucleus radius), then the screening is inessential and the amplitude coincides with that in the pure Coulomb field. At $1/\Delta \sim r_c$ the screening should be taken into account. Obviously, the impact parameters $\rho \gg r_c$ do not contribute to the total cross section. Due

to this fact we shall concentrate ourselves on the momentum transfer region corresponding to the impact parameter $\rho \leq r_c$. If $\mathbf{k}_{2\perp}^2/\omega_2 + \mathbf{k}_{3\perp}^2/\omega_3 \ll r_c^{-1}$, then it follows from (5) that the condition $\rho \leq r_c$ holds only when $|\mathbf{\Delta}_\perp| = |\mathbf{k}_{2\perp} + \mathbf{k}_{3\perp}| \geq r_c^{-1}$. Thus, the main contribution to the amplitude is given by the region of momentum transfer $\mathbf{\Delta}_\perp$, restricted from below. In this region $|\mathbf{\Delta}_\perp| \gg |\mathbf{\Delta}_\parallel|$, that is $\mathbf{\Delta} \approx \mathbf{\Delta}_\perp$. It is this region of parameters which we are going to consider. In addition, at $\omega/(m^2 + \tilde{\Delta}^2) \gg r_c$ the angles between $\mathbf{k}_{1,2,3}$ and $\mathbf{r}_{1,2,3}$ are either small or close to π , and corresponding expansions are used in our calculations.

According to the Furry theorem the photon-splitting amplitude is an odd function with respect to the parameter $Z\alpha$. Due to the singularity of the Coulomb potential in the momentum space ($-4\pi Z\alpha/\Delta^2$) the region of very small momentum transfers $\Delta \leq r_c^{-1}$ is essential only in the lowest (linear in $Z\alpha$) order of the perturbation theory. In this order the Weizsäcker-Williams method is applicable and the corresponding large logarithm appears in the cross section integrated over the angles of one of the final photons [16]. In next orders of perturbation theory with respect to the parameter $Z\alpha$ the integral over all momenta corresponding to the external field should be taken provided that their sum is equal to $\mathbf{\Delta}$. Therefore, even at $\Delta \sim r_c^{-1}$ each momentum is not small and the screening can be neglected. In the Born approximation the screening can be taken into account by multiplying the amplitude by the factor $[1 - F(\Delta^2)]$, where $F(\Delta^2)$ is the atomic electron form factor. Thus, to find the photon-splitting amplitude in a screened Coulomb field it is sufficient to solve the problem in the pure Coulomb field.

4 Green function

Let us pass now to the consideration of the Green function $D(\mathbf{r}, \mathbf{r}'|\varepsilon)$ appearing in eqs. (3) and (4). In Refs. [9, 10] the representation of this function has been found in the quasiclassical approximation for an arbitrary decreasing spherically symmetric potential. For the case of the Coulomb field, at small angle θ between vectors \mathbf{r} and $-\mathbf{r}'$ we obtain from eq. (14) of [9]:

$$D(\mathbf{r}, \mathbf{r}'|\varepsilon) = \frac{i e^{i\kappa(r+r')}}{4\pi\kappa r r'} \int_0^\infty dl \left[J_0(l\theta) + i Z\alpha \frac{(\boldsymbol{\alpha}, \mathbf{n} + \mathbf{n}')}{l\theta} J_1(l\theta) \right] \times \quad (6)$$

$$\exp \left[i \frac{l^2(r+r')}{2\kappa r r'} \right] \left(\frac{4\kappa^2 r r'}{l^2} \right)^{i Z\alpha\lambda},$$

where $\boldsymbol{\alpha} = \boldsymbol{\gamma}^0 \boldsymbol{\gamma}$, $\kappa^2 = \varepsilon^2 - m^2$, $\lambda = \varepsilon/\kappa$, $\mathbf{n} = \mathbf{r}/r$ and $\mathbf{n}' = \mathbf{r}'/r'$. Taking into account the relations

$$\int dl J_0(l\theta) g(l^2) = \frac{1}{2\pi} \int d\mathbf{q} \exp(i\mathbf{q}\boldsymbol{\theta}) g(q^2), \quad \frac{\boldsymbol{\theta}}{\theta} J_1(l\theta) = -\frac{1}{l} \frac{\partial}{\partial \boldsymbol{\theta}} J_0(l\theta),$$

where $g(l^2)$ is an arbitrary function and \mathbf{q} is a two-dimensional vector, one can rewrite eq. (6) in the form

$$D(\mathbf{r}, \mathbf{r}'|\varepsilon) = \frac{i e^{i\kappa(r+r')}}{8\pi^2 \kappa r r'} \int d\mathbf{q} \left[1 + Z\alpha \frac{\boldsymbol{\alpha}\mathbf{q}}{q^2} \right] \times \quad (7)$$

$$\exp \left[i \frac{q^2(r+r')}{2\kappa r r'} + i\mathbf{q}(\boldsymbol{\theta} + \boldsymbol{\theta}') \right] \left(\frac{4\kappa^2 r r'}{q^2} \right)^{i Z\alpha\lambda}.$$

Here $\boldsymbol{\theta} = \mathbf{r}_\perp/r$, $\boldsymbol{\theta}' = \mathbf{r}'_\perp/r'$. Expression (7) contains only elementary functions and the angles $\boldsymbol{\theta}$ and $\boldsymbol{\theta}'$ appear only in the factor $\exp[i\mathbf{q}(\boldsymbol{\theta} + \boldsymbol{\theta}')]$. Therefore, the representation (7) for the Green function is very convenient for calculations. At small angle between vectors \mathbf{r} and \mathbf{r}' , for the case of pure Coulomb field, we get from eq. (15) of [9]:

$$D(\mathbf{r}, \mathbf{r}'|\varepsilon) = -\frac{e^{i\kappa|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} \left(\frac{r}{r'} \right)^{i Z\alpha\lambda \text{sign}(r-r')}. \quad (8)$$

One can see that in this case the Green function differs from that of free Dirac equation only by the phase factor. It is easy to check that after the substitution of the expressions (7) and (8) for the Green

functions to the splitting amplitudes (3) and (4) all phase factors of the form $r^{\pm iZ\alpha}$ cancel. Note that one can replace κ by $|\varepsilon| - m^2/(2|\varepsilon|)$ in eqs. (7) and (8). Moreover, one should take the relativistic correction $m^2/(2|\varepsilon|)$ into account only in the factor $\exp[i\kappa(r + r')]$.

5 Calculation of the amplitudes $M^{(3)}$ and $M^{(2)}$

Consider now the diagrams, containing three Green function (See Fig. 1). Obviously, the contribution of diagram (b) can be obtained from the contribution of diagram (a) by replacing $Z\alpha \rightarrow -Z\alpha$ and changing the overall sign. It provides the fulfillment of the Furry theorem: the sum of two contributions (a) and (b) in Fig. 1 is odd function of $Z\alpha$. Therefore, this sum is equal to the odd in $Z\alpha$ part of diagram (a) multiplied by two. Then, at the calculation of diagram (a) the region of integration over z_3 is divided into two: $z_3 > 0$ (photon with the momentum \mathbf{k}_3 is ahead) and $z_3 < 0$ (photon is behind). In the first region the angles between vectors \mathbf{r}_2 , \mathbf{r}_3 and $-\mathbf{r}_1$ are small. In the second region the angles between \mathbf{r}_1 , \mathbf{r}_3 and $-\mathbf{r}_2$ are small. Denote the contribution of the first region to the diagram (a) as $M_1^{(3)}$ and the contribution of the second region as $M_2^{(3)}$. Introduce vectors $\mathbf{f}_2 = \mathbf{k}_{2\perp}/\omega_2$ and $\mathbf{f}_3 = \mathbf{k}_{3\perp}/\omega_3$ ($|\mathbf{f}_{2,3}| \ll 1$), as well as $\boldsymbol{\theta}_i = \mathbf{r}_{i\perp}/r_i = \mathbf{n}_{i\perp}$ ($i=1,2,3$). Taking into account the smallness of the angles we have $d\mathbf{r}_i = r_i^2 dr_i d\boldsymbol{\theta}_i$. It is convenient to perform further calculations in terms of the helicity amplitudes $M_{\lambda_1\lambda_2\lambda_3}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$. It is sufficient to calculate three of them, for instance, $M_{+--}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$, $M_{+++}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$ and $M_{++-}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$. The others can be obtained by substitutions. Note that within the accuracy of our calculations there is no need to take into account the corrections to the transverse part of the polarization vectors $\mathbf{e}_{2,3}$. Due to the relation $\mathbf{e}\mathbf{k} = 0$ the z -component of the polarization vectors can be expressed via the transverse one as $(\mathbf{e}_{2,3})_z = -\mathbf{e}_{2,3}\mathbf{f}_{2,3}$. So, for a given helicity one can put the transverse part of the final photon polarization vector to be equal to the polarization vector of the photon with the same helicity propagating along z axis. Further the polarization vector \mathbf{e}_+ corresponding to the positive helicity is denoted as \mathbf{e} . Then the negative helicity polarization vector \mathbf{e}_- is equal to \mathbf{e}^* . Note that to obtain M_{++-} in our approach it is necessary to calculate both $M_{++-}^{(3)}$ and $M_{+-+}^{(3)}$ since the arrangement of the vertices on the diagrams is space ordered.

Let us substitute the expressions (7) and (8) to (3) and make the obvious expansion at small angles, taking into account terms, quadratic in \mathbf{f}_i and $\boldsymbol{\theta}_i$. Introduce notations $\kappa_2 = \omega_2 - \varepsilon$, $\kappa_3 = \omega_3 + \varepsilon$ and pass to the variables

$$\mathbf{q}_2 \rightarrow \kappa_2 \mathbf{q}_2, \quad \mathbf{q}_3 \rightarrow \kappa_3 \mathbf{q}_3, \quad R_1 = \frac{\omega_1}{\kappa_2 \kappa_3} r_1, \quad R_2 = \frac{\omega_2}{\varepsilon \kappa_2} r_2, \quad R_3 = \frac{\omega_3}{\varepsilon \kappa_3} r_3.$$

Simple integration over $\boldsymbol{\theta}_i$ leads to

$$M_1^{(3)} = \frac{e^3}{32\pi^3 \omega_1 \omega_2 \omega_3} \int_0^{\omega_2} \varepsilon \kappa_2 \kappa_3 d\varepsilon \int_0^\infty dR_1 \int_0^\infty dR_2 \int_0^L \frac{dR_3}{R_1 R} \iint d\mathbf{q}_2 d\mathbf{q}_3 \left(\frac{q_2}{q_3}\right)^{2iZ\alpha} e^{i\Phi} T, \quad (9)$$

where $L = R_2 \omega_3 \kappa_2 / \omega_2 \kappa_3$, and

$$T = \frac{1}{4} \text{Tr} \left\{ \left(1 + \frac{Z\alpha \boldsymbol{\alpha} \mathbf{q}_3}{\kappa_3 q_3^2} \right) \left(\frac{2}{R_1} \mathbf{e}_1 \mathbf{Q} - \hat{\varepsilon}_1 \hat{k}_1 \right) \left(1 - \frac{Z\alpha \boldsymbol{\alpha} \mathbf{q}_2}{\kappa_2 q_2^2} \right) \times \right. \\ \left. \left[\left(\frac{2}{R} (\mathbf{e}_2^*, \mathbf{Q} + \varepsilon R_3 \mathbf{f}_{23}) - \varepsilon_2^* \hat{k}_2 \right) \left(\frac{2}{R} (\mathbf{e}_3^*, \mathbf{Q} + \varepsilon R_2 \mathbf{f}_{23}) - \varepsilon_3^* \hat{k}_3 \right) - \frac{4i}{R} \mathbf{e}_2^* \mathbf{e}_3^* \right] \right\}, \quad (10)$$

$$\Phi = \left[\left(\frac{1}{R} + \frac{1}{R_1} \right) \frac{\mathbf{Q}^2}{2} + \frac{\varepsilon^2 R_2 R_3 \mathbf{f}_{23}^2}{2R} - \frac{(\kappa_2 \mathbf{q}_2 - \kappa_3 \mathbf{q}_3, \boldsymbol{\Delta})}{\omega_1} - \right. \\ \left. \frac{(\omega_3 \kappa_2 R_2 - \omega_2 \kappa_3 R_3)}{\omega_1 R} (\mathbf{Q} \mathbf{f}_{23}) - \frac{m^2}{2} (R_1 + R) \right],$$

$$R = R_2 - R_3, \quad \mathbf{f}_{23} = \mathbf{f}_2 - \mathbf{f}_3, \quad \mathbf{Q} = \mathbf{q}_2 + \mathbf{q}_3, \quad \boldsymbol{\Delta} = \omega_2 \mathbf{f}_2 + \omega_3 \mathbf{f}_3.$$

It is convenient to transform the function T in eq. (10) to the form which does not contain the parameter $Z\alpha$. To this purpose we use the identities

$$Z\alpha \frac{\mathbf{q}_2}{q_2^2} \left(\frac{q_2}{q_3} \right)^{2iZ\alpha} = -\frac{i}{2} \frac{\partial}{\partial \mathbf{q}_2} \left(\frac{q_2}{q_3} \right)^{2iZ\alpha},$$

$$Z\alpha \frac{\mathbf{q}_3}{q_3^2} \left(\frac{q_2}{q_3} \right)^{2iZ\alpha} = \frac{i}{2} \frac{\partial}{\partial \mathbf{q}_3} \left(\frac{q_2}{q_3} \right)^{2iZ\alpha},$$

and integrate by parts over \mathbf{q}_2 and \mathbf{q}_3 in (9). After that some terms in the function T are independent of the variable R_1 and others contain it in the form of factors $1/R_1, 1/R_1^2$. Taking the trace and integrating by parts over R_1 the terms with the factor $1/R_1^2$, we obtain for different polarizations:

$$T_{+--} = \frac{8}{R_1 R^2} (\mathbf{e}\mathbf{Q})(\mathbf{e}\mathbf{Q}_2)(\mathbf{e}\mathbf{Q}_3); \quad (11)$$

$$T_{+++} = -\frac{4}{R_1 R^2} \left(\frac{\kappa_2}{\kappa_3} + \frac{\kappa_3}{\kappa_2} \right) (\mathbf{e}\mathbf{Q})(\mathbf{e}^*\mathbf{Q}_2)(\mathbf{e}^*\mathbf{Q}_3) - \frac{\omega_1}{\varepsilon R^3} \left(\mathbf{e}^*, \frac{\omega_2}{\kappa_2} \mathbf{Q}_3^2 \mathbf{Q}_2 - \frac{\omega_3}{\kappa_3} \mathbf{Q}_2^2 \mathbf{Q}_3 \right) +$$

$$\frac{2i\omega_1^2}{\kappa_2 \kappa_3 R^2} (\mathbf{e}^*, \mathbf{Q}_2 + \mathbf{Q}_3) + \frac{m^2 \omega_1}{\varepsilon R} \left(\mathbf{e}^*, \frac{\omega_2}{\kappa_2} \mathbf{Q}_2 - \frac{\omega_3}{\kappa_3} \mathbf{Q}_3 \right);$$

$$T_{++-} = -\frac{4}{R_1 R^2} \left(\frac{\kappa_2}{\varepsilon} + \frac{\varepsilon}{\kappa_2} \right) (\mathbf{e}\mathbf{Q})(\mathbf{e}\mathbf{Q}_2)(\mathbf{e}^*\mathbf{Q}_3) +$$

$$\frac{\omega_2 \omega_3}{\varepsilon \kappa_3 R_1 R^2} (\mathbf{Q}_2^2 + \mathbf{Q}_3^2)(\mathbf{e}\mathbf{Q}) - \frac{\omega_1 \omega_2}{\kappa_2 \kappa_3 R} \left(\frac{\mathbf{Q}_3^2}{R^2} - m^2 \right) (\mathbf{e}\mathbf{Q}_2) +$$

$$\frac{4i}{R_1 R} \left(\frac{\omega_1 \omega_2}{\kappa_2 \kappa_3} - 2 \right) (\mathbf{e}\mathbf{Q}) + \frac{2i\omega_1 \omega_2}{\kappa_2 \kappa_3 R^2} (\mathbf{e}, \mathbf{Q}_2 + \mathbf{Q}_3),$$

where $\mathbf{Q}_2 = \mathbf{Q} + \varepsilon R_2 \mathbf{f}_{23}$ and $\mathbf{Q}_3 = \mathbf{Q} + \varepsilon R_3 \mathbf{f}_{23}$. The function T_{+--} can be obtained from T_{++-} by the substitution $\omega_2 \leftrightarrow \omega_3, \kappa_2 \leftrightarrow \kappa_3, \mathbf{Q}_2 \leftrightarrow \mathbf{Q}_3$ and $\varepsilon \rightarrow -\varepsilon$.

In the same way for the contribution $M_2^{(3)}$ we obtain:

$$M_2^{(3)} = \frac{e^3}{32\pi^3 \omega_1 \omega_2 \omega_3} \int_0^{\omega_2} \varepsilon \kappa_2 \kappa_3 d\varepsilon \int_0^\infty dR_1 \int_0^\infty dR_2 \int_0^{L_1} \frac{dR_3}{r R_2} \iint d\mathbf{q}_2 d\mathbf{q}_3 \left(\frac{q_2}{q_3} \right)^{2iZ\alpha} e^{i\tilde{\Phi}} \tilde{T}, \quad (12)$$

where $L_1 = R_1 \omega_3 \kappa_2 / \omega_1 \varepsilon, r = R_1 + R_3,$

$$\tilde{\Phi} = \left[\left(\frac{1}{r} + \frac{1}{R_2} \right) \frac{\mathbf{Q}^2}{2} - \frac{\kappa_3^2 R_1 R_3 \mathbf{f}_3^2}{2r} - \frac{(\kappa_2 \mathbf{q}_2 - \varepsilon \mathbf{q}_3, \mathbf{\Delta})}{\omega_2} + \right.$$

$$\left. \frac{(\omega_3 \kappa_2 R_1 - \varepsilon \omega_1 R_3)}{\omega_2 r} (\mathbf{Q}\mathbf{f}_3) - \frac{m^2}{2} (R_2 + r) \right], \quad (13)$$

and the function \tilde{T} for different polarizations reads

$$\tilde{T}_{+--} = -\frac{8}{r^2 R_2} (\mathbf{e}\mathbf{Q})(\mathbf{e}\mathbf{P}_1)(\mathbf{e}\mathbf{P}_3); \quad (14)$$

$$\tilde{T}_{+++} = \frac{4}{r^2 R_2} \left(\frac{\kappa_2}{\varepsilon} + \frac{\varepsilon}{\kappa_2} \right) (\mathbf{e}^*\mathbf{Q})(\mathbf{e}\mathbf{P}_1)(\mathbf{e}\mathbf{P}_3) + \frac{\omega_2}{\kappa_3 r^3} \left(\mathbf{e}, \frac{\omega_1}{\kappa_2} \mathbf{P}_3^2 \mathbf{P}_1 + \frac{\omega_3}{\varepsilon} \mathbf{P}_1^2 \mathbf{P}_3 \right) -$$

$$\frac{2i\omega_2^2}{\kappa_2 \varepsilon r^2} (\mathbf{e}, \mathbf{P}_1 + \mathbf{P}_3) - \frac{m^2 \omega_2}{\kappa_3 r} \left(\mathbf{e}, \frac{\omega_1}{\kappa_2} \mathbf{P}_1 + \frac{\omega_3}{\varepsilon} \mathbf{P}_3 \right);$$

$$\tilde{T}_{++-} = \frac{4}{r^2 R_2} \left(\frac{\kappa_2}{\kappa_3} + \frac{\kappa_3}{\kappa_2} \right) (\mathbf{e}^*\mathbf{Q})(\mathbf{e}^*\mathbf{P}_1)(\mathbf{e}\mathbf{P}_3) +$$

$$\frac{\omega_1 \omega_3}{\varepsilon \kappa_3 r^2 R_2} (\mathbf{P}_1^2 + \mathbf{P}_3^2)(\mathbf{e}^*\mathbf{Q}) + \frac{\omega_1 \omega_2}{\kappa_2 \varepsilon r} \left(\frac{\mathbf{P}_3^2}{r^2} - m^2 \right) (\mathbf{e}^*\mathbf{P}_1) -$$

$$\frac{4i}{r R_2} \left(\frac{\omega_1 \omega_2}{\kappa_2 \varepsilon} - 2 \right) (\mathbf{e}^*\mathbf{Q}) - \frac{2i\omega_1 \omega_2}{\kappa_2 \varepsilon r^2} (\mathbf{e}^*, \mathbf{P}_1 + \mathbf{P}_3),$$

where $\mathbf{P}_1 = \mathbf{Q} + \kappa_3 R_1 \mathbf{f}_3$ and $\mathbf{P}_3 = \mathbf{Q} - \kappa_3 R_3 \mathbf{f}_3$. One can get the function \tilde{T}_{+-+} from \tilde{T}_{+++} by the substitution $\omega_1 \leftrightarrow -\omega_3$, $\kappa_2 \leftrightarrow -\varepsilon$, $\mathbf{P}_1 \leftrightarrow \mathbf{P}_3$ and $\mathbf{e} \leftrightarrow \mathbf{e}^*$. Note that the integrand in eqs. (12) and (14) for helicity amplitudes $M_2^{(3)}$ can be obtained from the integrand in eqs. (9), (11) for $M_1^{(3)}$ by means of substitutions

$$\begin{aligned} \mathbf{q}_{2,3} &\rightarrow -\mathbf{q}_{2,3} \quad , \quad \omega_1 \leftrightarrow \omega_2 \quad , \quad \omega_3 \rightarrow -\omega_3 \quad , \quad \kappa_3 \leftrightarrow \varepsilon \quad , \\ R_1 &\leftrightarrow R_2 \quad , \quad R_3 \rightarrow -R_3 \quad , \quad \mathbf{f}_{23} \leftrightarrow -\mathbf{f}_3 \quad , \quad \mathbf{f}_2 \rightarrow -\mathbf{f}_2 . \end{aligned} \quad (15)$$

so, that

$$\begin{aligned} T_{+--} &\rightarrow \tilde{T}_{+--} \quad , \quad T_{+-+} \rightarrow \tilde{T}_{+-+} \quad , \\ T_{+++} &\rightarrow \tilde{T}_{+++} (\mathbf{e} \leftrightarrow \mathbf{e}^*) \quad , \quad T_{++-} \rightarrow \tilde{T}_{++-} (\mathbf{e} \leftrightarrow \mathbf{e}^*) . \end{aligned}$$

As to the amplitude $M^{(2)}$, after the integration over the angles θ_i , we get:

$$\begin{aligned} M_{+--}^{(2)} &= 0 \quad , \quad M_{+++}^{(2)} = (\mathbf{e}^* , \mathbf{M}_{12} + \mathbf{M}_{13}) \quad , \\ M_{++-}^{(2)} &= (\mathbf{e} , \mathbf{M}_{12} + \mathbf{M}_{23}) \quad , \quad M_{+-+}^{(2)} = (\mathbf{e} , \mathbf{M}_{13} + \mathbf{M}_{23}) , \end{aligned} \quad (16)$$

where

$$\begin{aligned} \mathbf{M}_{23} &= -\frac{ie^3}{16\pi^3} \int_{-\omega_3}^{\omega_2} d\varepsilon \int_0^\infty \frac{dR_1}{R_1^2} \int_0^\infty \frac{dR_2}{R_2^2} \left[R_1 + \left(\frac{\kappa_2 - \kappa_3}{\omega_1} \right)^2 R_2 \right] \\ &\quad \iint d\mathbf{q}_2 d\mathbf{q}_3 \mathbf{Q} \left(\frac{q_2}{q_3} \right)^{2iZ\alpha} \exp \left\{ i \left[\left(\frac{1}{R_1} + \frac{1}{R_2} \right) \frac{\mathbf{Q}^2}{2} + \right. \right. \\ &\quad \left. \left. \frac{\omega_2 \omega_3 \kappa_2 \kappa_3}{2\omega_1^2} \mathbf{f}_{23}^2 R_2 - \frac{(\kappa_2 \mathbf{q}_2 - \kappa_3 \mathbf{q}_3, \Delta)}{\omega_1} - \frac{m^2}{2} (R_1 + R_2) \right] \right\} ; \\ \mathbf{M}_{13} &= \frac{ie^3}{16\pi^3} \int_0^{\omega_2} d\varepsilon \int_0^\infty \frac{dR_1}{R_1^2} \int_0^\infty \frac{dR_2}{R_2^2} \left[R_2 + \left(\frac{\kappa_2 - \varepsilon}{\omega_2} \right)^2 R_1 \right] \\ &\quad \iint d\mathbf{q}_2 d\mathbf{q}_3 \mathbf{Q} \left(\frac{q_2}{q_3} \right)^{2iZ\alpha} \exp \left\{ i \left[\left(\frac{1}{R_1} + \frac{1}{R_2} \right) \frac{\mathbf{Q}^2}{2} - \right. \right. \\ &\quad \left. \left. \frac{\omega_1 \omega_3 \varepsilon \kappa_2}{2\omega_2^2} \mathbf{f}_3^2 R_1 - \frac{(\kappa_2 \mathbf{q}_2 - \varepsilon \mathbf{q}_3, \Delta)}{\omega_2} - \frac{m^2}{2} (R_1 + R_2) \right] \right\} , \end{aligned} \quad (17)$$

and the vector \mathbf{M}_{12} can be obtained from \mathbf{M}_{13} by the substitutions $\omega_2 \leftrightarrow \omega_3$ and $\mathbf{f}_3 \leftrightarrow \mathbf{f}_2$. As we shall see, many terms cancel out in the sum $M^{(2)} + M^{(3)}$.

In general case further transformations of the formulae obtained leads to four-fold integral with the integrand containing the elementary functions and this problem will be considered in detail elsewhere. In what follows we restrict ourselves to the case $|\mathbf{k}_{2\perp}| = |\omega_2 \mathbf{f}_2| \gg m$, $|\mathbf{k}_{3\perp}| = |\omega_3 \mathbf{f}_3| \gg m$ when the amplitudes can be essentially simplified. This range of the parameters corresponds to large virtuality of the electron-positron pair compared to the electron mass, which in this case can be neglected. Note that the ratio between the momentum transfer $\Delta = |\Delta|$ and the electron mass m can be arbitrary, since Δ determines a typical impact parameter $\rho \sim 1/\Delta$ rather than pair virtuality.

6 Zero mass limit

It is easy to see that putting $m = 0$ in the expressions obtained leads to the logarithmic divergences in some terms (in other words, these terms contain $\log m$ at finite mass). For instance, one obtains such logarithm in the amplitude $M_1^{(3)}$ integrating over R_1 the terms in T which do not contain the factor $1/R_1$ (see (11)). The final result, as it should be, does not contain the logarithms of mass. But the cancellation of these logarithms between different terms is rather tricky.

Taking the integral over R_1 in (9) for $T = T_{+--}$ we do not obtain any logarithm in $M_{+--}^{(3)}$ while $M_{+--}^{(2)} = 0$. Eq. (11) for T_{+++} and T_{++-} contains the terms proportional to \mathbf{Q}_2^2 . It is convenient to pass in these terms from the variables R_2 and R_3 to R_2 and $y = R_3/R$, and then integrate by parts over y . In the terms containing \mathbf{Q}_3^2 we pass to the variables R_3 and $y = R_3/R$ and also integrate by parts over y . As a result, in the two-fold integral over R_2 and R_3 all logarithms cancel out and one can put $m = 0$. After that T_{+++} and T_{++-} are transformed to

$$\begin{aligned} T_{+++} &= -\frac{4}{R_1 R^2} \left(\frac{\kappa_2}{\kappa_3} + \frac{\kappa_3}{\kappa_2} \right) (\mathbf{e}\mathbf{Q})(\mathbf{e}^*\mathbf{Q}_2)(\mathbf{e}^*\mathbf{Q}_3) \quad ; \\ T_{++-} &= -\frac{4}{R_1 R^2} \left(\frac{\kappa_2}{\varepsilon} + \frac{\varepsilon}{\kappa_2} \right) (\mathbf{e}\mathbf{Q})[(\mathbf{e}\mathbf{Q}_2)(\mathbf{e}^*\mathbf{Q}_3) - iR] \quad . \end{aligned} \quad (18)$$

In addition, there are integrated terms at $y = \omega_3 \kappa_2 / \omega_1 \varepsilon$ (upper limit) and $y = 0$ (lower limit). Making the similar transformations for the amplitude $M_2^{(3)}$ we obtain that \tilde{T}_{+++} and \tilde{T}_{++-} turn to

$$\begin{aligned} \tilde{T}_{++-} &= \frac{4}{r^2 R_2} \left(\frac{\kappa_2}{\varepsilon} + \frac{\varepsilon}{\kappa_2} \right) (\mathbf{e}^*\mathbf{Q})(\mathbf{e}\mathbf{P}_1)(\mathbf{e}\mathbf{P}_3) \quad ; \\ \tilde{T}_{+++} &= \frac{4}{r^2 R_2} \left(\frac{\kappa_2}{\kappa_3} + \frac{\kappa_3}{\kappa_2} \right) (\mathbf{e}^*\mathbf{Q})[(\mathbf{e}^*\mathbf{P}_1)(\mathbf{e}\mathbf{P}_3) - iR] \quad . \end{aligned} \quad (19)$$

The integrated terms corresponding to the lower limit cancel out in the sum of $M_1^{(3)}$ and $M_2^{(3)}$. Remind that to calculate the amplitude $M^{(3)}$ we have to find the sum $M_1^{(3)} + (\mathbf{k}_2 \leftrightarrow \mathbf{k}_3, \mathbf{e}_2 \leftrightarrow \mathbf{e}_3)$, extract the odd in $Z\alpha$ part and multiply the result by two. After that the contribution of the upper-limit integrated terms of $M_1^{(3)}$ vanishes in the case M_{+++} , and in the case M_{++-} and M_{+-+} gives the finite result after the summation with $(\mathbf{e}\mathbf{M}_{23})$. To cancel the logarithmic terms we exploited the antisymmetry of some integrands with respect to the substitution $\varepsilon \rightarrow \omega_2 - \omega_3 - \varepsilon$, $\mathbf{q}_2 \leftrightarrow -\mathbf{q}_3$. The contribution of the upper-limit integrated terms in $M_2^{(3)}$ vanishes in the case of M_{++-} , and for M_{+++} and M_{+-+} gives the finite result at $m = 0$ after the summation with $(\mathbf{e}^*\mathbf{M}_{13})$ and $(\mathbf{e}\mathbf{M}_{13})$, respectively. Analogously, the amplitudes $(\mathbf{e}^*\mathbf{M}_{12})$ and $(\mathbf{e}\mathbf{M}_{12})$ cancel singular terms of $M_2^{(3)}(\mathbf{k}_2 \leftrightarrow \mathbf{k}_3)$ for the amplitudes M_{+++} and M_{++-} . For the amplitude M_{+-+} the upper-limit integrated terms of $M_2^{(3)}(\mathbf{k}_2 \leftrightarrow \mathbf{k}_3)$ cancel out. As a result, the sum of integrated terms and $M^{(2)}$ gives the additional contributions to helicity amplitudes. We represent them in the form:

$$\delta M = -\frac{e^3}{4\pi^3} \int_0^\infty \frac{dR}{R} \iint \frac{d\mathbf{q}_2 d\mathbf{q}_3}{\mathbf{Q}^2} \left[\left(\frac{q_2}{q_3} \right)^{2iZ\alpha} - \left(\frac{q_3}{q_2} \right)^{2iZ\alpha} \right] F \quad . \quad (20)$$

For different polarizations the function F reads

$$\begin{aligned} F_{+--} &= 0 \quad ; \quad F_{+-+} = (\mathbf{e}\mathbf{Q}) \left[\int_{-\omega_3}^{\omega_2} d\varepsilon \frac{\kappa_2 \kappa_3^2}{\omega_1^2 \varepsilon} e^{i\psi_1} - \int_0^{\omega_2} d\varepsilon \frac{\kappa_2 \varepsilon^2}{\omega_2^2 \kappa_3} e^{i\psi_2} \right] ; \\ F_{+++} &= (\mathbf{e}^*\mathbf{Q}) \int_0^{\omega_2} d\varepsilon \frac{\varepsilon \kappa_2^2}{\omega_2^2 \kappa_3} e^{i\psi_2} + (\omega_2 \leftrightarrow \omega_3, \mathbf{f}_2 \leftrightarrow \mathbf{f}_3) \quad ; \\ F_{++-} &= F_{+-+}(\omega_2 \leftrightarrow \omega_3, \mathbf{f}_2 \leftrightarrow \mathbf{f}_3) \quad ; \end{aligned} \quad (21)$$

$$\psi_1 = \frac{\mathbf{Q}^2}{2R} + \frac{\omega_2 \omega_3 \kappa_2 \kappa_3}{2\omega_1^2} \mathbf{f}_{23}^2 R - \frac{(\kappa_2 \mathbf{q}_2 - \kappa_3 \mathbf{q}_3, \mathbf{\Delta})}{\omega_1} \quad .$$

The phase ψ_2 can be obtained from ψ_1 by substitution (15). To obtain eq. (20) we have integrated over one of the radii. Note that the singularity of the integrand in (21) at $\varepsilon = 0$ disappears in the total expression for the amplitude of the process so that no regularization is required.

Putting $m = 0$ in phases Φ and $\tilde{\Phi}$, we take elementary integrals over R_1 in (9) and over R_2 in (12). After passing from the variables q_2 and q_3 to $\mathbf{Q} = \mathbf{q}_2 + \mathbf{q}_3$ and $\mathbf{q} = \mathbf{q}_2 - \mathbf{q}_3$, the integral with respect to

\mathbf{q} reads:

$$J = \int \frac{d\mathbf{q}}{\mathbf{Q}^2} \left(\frac{|\mathbf{q} + \mathbf{Q}|}{|\mathbf{q} - \mathbf{Q}|} \right)^{2iZ\alpha} \exp\left(-\frac{i}{2}\mathbf{q}\mathbf{\Delta}\right) . \quad (22)$$

Let us change the variable $\mathbf{q} \rightarrow |\mathbf{Q}|\mathbf{q}$ and represent \mathbf{Q} and $\mathbf{\Delta}$ as $\mathbf{Q} = |\mathbf{Q}|\boldsymbol{\lambda}_1$ and $\mathbf{\Delta} = |\mathbf{\Delta}|\boldsymbol{\lambda}_2$, respectively. After that it is easy to see that J depends on $S = (\boldsymbol{\lambda}_1\boldsymbol{\lambda}_2)$ and $|\mathbf{Q}||\mathbf{\Delta}|$. Note that in two-dimensional case the form $P = \epsilon_{ij}\lambda_1^i\lambda_2^j$ is also invariant with respect to rotations. However, the even powers of P can be expressed via S ($P^2 = 1 - S^2$), and the odd powers of P change their sign after reflection. On the other hand, J is invariant under reflection. It becomes obvious if the reflection of \mathbf{q} is made along with the reflection of $\boldsymbol{\lambda}_1$ and $\boldsymbol{\lambda}_2$. Therefore, J is invariant with respect to the change of variables $\mathbf{Q} \leftrightarrow \mathbf{\Delta}$. Therefore, we can represent J as

$$J = \int \frac{d\mathbf{q}}{\mathbf{\Delta}^2} \left(\frac{|\mathbf{q} + \mathbf{\Delta}|}{|\mathbf{q} - \mathbf{\Delta}|} \right)^{2iZ\alpha} \exp\left(-\frac{i}{2}\mathbf{q}\mathbf{Q}\right) . \quad (23)$$

This form is very convenient for further calculations. Now one can easily take the integrals over \mathbf{Q} and all radii. Summing all contributions we finally get:

$$\begin{aligned} M &= \frac{8\epsilon^3}{\pi^2\omega_1\omega_2\omega_3\mathbf{\Delta}^2} \int d\mathbf{q} (\mathbf{T}\nabla_{\mathbf{q}}) \text{Im} \left(\frac{|\mathbf{q} + \mathbf{\Delta}|}{|\mathbf{q} - \mathbf{\Delta}|} \right)^{2iZ\alpha} ; \quad (24) \\ \mathbf{T}_{+--} &= \omega_3 e \int_0^{\omega_2} d\varepsilon \frac{\kappa_2^2}{(\mathbf{e}^*\mathbf{a})} \left[\frac{(\mathbf{e}\mathbf{b})\kappa_3}{\omega_1\mathcal{D}_1} - \frac{(\mathbf{e}\mathbf{c})\varepsilon}{\omega_2\mathcal{D}_3} \right] + \left(\begin{array}{l} \omega_2 \leftrightarrow \omega_3 \\ \mathbf{f}_2 \leftrightarrow \mathbf{f}_3 \end{array} \right) ; \\ \mathbf{T}_{+++} &= \omega_3 \int_0^{\omega_2} d\varepsilon \kappa_2 \left[e^* \frac{\varepsilon}{\omega_2\mathcal{D}_3} \left(\frac{\kappa_3 - \kappa_2}{2} + \frac{(\mathbf{e}^*\mathbf{f}_3)}{(\mathbf{e}^*\mathbf{a})} (\kappa_2^2 + \kappa_3^2) \right) - \right. \\ &\quad \left. e \frac{(\mathbf{e}^*\mathbf{b})(\kappa_2^2 + \kappa_3^2)}{2(\mathbf{e}\mathbf{a})\omega_1\mathcal{D}_1} \right] + \left(\begin{array}{l} \omega_2 \leftrightarrow \omega_3 \\ \mathbf{f}_2 \leftrightarrow \mathbf{f}_3 \end{array} \right) ; \\ \mathbf{T}_{++-} &= \omega_3 \int_0^{\omega_2} d\varepsilon \kappa_2 \left[e \frac{\kappa_3}{\omega_1\mathcal{D}_1} \left(\frac{\kappa_2 - \varepsilon}{2} - \frac{(\mathbf{e}\mathbf{f}_{23})}{(\mathbf{e}\mathbf{a})} (\kappa_2^2 + \varepsilon^2) \right) + \right. \\ &\quad \left. e^* \frac{(\mathbf{e}\mathbf{c})(\kappa_2^2 + \varepsilon^2)}{2(\mathbf{e}^*\mathbf{a})\omega_2\mathcal{D}_3} \right] + \omega_2 e \int_{-\omega_3}^0 d\varepsilon \kappa_3 \left[\frac{\kappa_3(\kappa_2^2 + \varepsilon^2)}{(\mathbf{e}^*\mathbf{b})} \left(\frac{(\mathbf{e}^*\mathbf{f}_{23})}{\omega_1\mathcal{D}_1} + \frac{(\mathbf{e}^*\mathbf{f}_2)}{\omega_3\mathcal{D}_2} \right) - \right. \\ &\quad \left. \frac{\varepsilon\kappa_3 + \kappa_2^2}{2\omega_1\mathcal{D}_1} + \frac{\varepsilon^2 - \kappa_2\kappa_3}{2\omega_3\mathcal{D}_2} \right] . \end{aligned}$$

Obviously, \mathbf{T}_{+--} can be obtained from \mathbf{T}_{++-} by the substitutions $\omega_2 \leftrightarrow \omega_3$, $\mathbf{f}_2 \leftrightarrow \mathbf{f}_3$. In eq. (24) the following notation is introduced:

$$\begin{aligned} \mathcal{D}_1 &= \frac{\omega_2\kappa_3\mathbf{a}^2 - \omega_3\kappa_2\mathbf{b}^2}{\omega_1\varepsilon} - i0 , \quad \mathcal{D}_2 = \frac{\omega_2\kappa_3\tilde{\mathbf{c}}^2 - \omega_1\varepsilon\mathbf{b}^2}{\omega_3\kappa_2} , \\ \mathcal{D}_3 &= \frac{\omega_1\varepsilon\mathbf{a}^2 + \omega_3\kappa_2\mathbf{c}^2}{\omega_2\kappa_3} , \\ \mathbf{a} &= \mathbf{q} - \mathbf{\Delta} + 2\kappa_2\mathbf{f}_2 , \quad \mathbf{b} = \mathbf{q} + \mathbf{\Delta} - 2\kappa_3\mathbf{f}_3 , \\ \mathbf{c} &= \mathbf{q} + \mathbf{\Delta} - 2\varepsilon\mathbf{f}_{23} , \quad \tilde{\mathbf{c}} = \mathbf{q} - \mathbf{\Delta} + 2\varepsilon\mathbf{f}_{23} . \end{aligned} \quad (25)$$

At the derivation of (24) we used the identity

$$\mathbf{Q} \exp\left(-\frac{i}{2}\mathbf{q}\mathbf{Q}\right) = 2i\nabla_{\mathbf{q}} \exp\left(-\frac{i}{2}\mathbf{q}\mathbf{Q}\right)$$

and integrated by parts over \mathbf{q} . Note that vectors \mathbf{e} and \mathbf{e}^* appeared in denominators in (24) owing to the application of the relation $2(\mathbf{e}\mathbf{a})(\mathbf{e}^*\mathbf{a}) = \mathbf{a}^2$.

7 Asymptotics at small Δ

In the small-angle approximation ($|\mathbf{f}_2|, |\mathbf{f}_3| \ll 1$) the cross section of the process reads:

$$d\sigma = \frac{\omega_1^2}{2^8 \pi^5} |M|^2 x(1-x) dx d\mathbf{f}_2 d\mathbf{f}_3 \quad , \quad (26)$$

where $x = \omega_2/\omega_1$, so that $\omega_3 = \omega_1(1-x)$. Let us define $\boldsymbol{\rho} = (\omega_2 \mathbf{f}_2 - \omega_3 \mathbf{f}_3)/2$. In terms of the variables $\boldsymbol{\rho}$ and Δ the cross section has the form

$$d\sigma = |M|^2 \frac{d\Delta d\boldsymbol{\rho} dx}{2^8 \pi^5 \omega_1^2 x(1-x)} \quad , \quad (27)$$

Consider the asymptotics of the amplitudes at $|\Delta| \ll |\boldsymbol{\rho}|$. To this purpose multiply \mathbf{T} in (24) by

$$1 = \vartheta(q_0^2 - \mathbf{q}^2) + \vartheta(\mathbf{q}^2 - q_0^2) \quad ,$$

where $|\Delta| \ll q_0 \ll |\boldsymbol{\rho}|$. Then, for the term in (24) proportional to $\vartheta(q_0^2 - \mathbf{q}^2)$ one can put $\mathbf{q} = 0$ and $\Delta = 0$ in \mathbf{T} and integrate by parts over \mathbf{q} . After that, using the relation $\nabla_{\mathbf{q}} \vartheta(q_0^2 - \mathbf{q}^2) = -2\mathbf{q} \delta(q_0^2 - \mathbf{q}^2)$ one can easily take the integral over \mathbf{q} since at $|\mathbf{q}| = q_0 \gg |\Delta|$ one has

$$\text{Im} \left(\frac{|\mathbf{q} + \Delta|}{|\mathbf{q} - \Delta|} \right)^{2iZ\alpha} \approx 4Z\alpha \frac{\mathbf{q}\Delta}{\mathbf{q}^2} \quad .$$

As a result, in the region $|\mathbf{q}| < q_0$ the term proportional to $Z\alpha$ is independent of q_0 and the terms of next orders in $Z\alpha$ are small in the parameter $|\Delta|/q_0$.

For the term proportional to $\vartheta(\mathbf{q}^2 - q_0^2)$ we get

$$\nabla_{\mathbf{q}} \text{Im} \left(\frac{|\mathbf{q} + \Delta|}{|\mathbf{q} - \Delta|} \right)^{2iZ\alpha} \approx 4Z\alpha \frac{\mathbf{q}^2 \Delta - 2\mathbf{q}(\mathbf{q}\Delta)}{|\mathbf{q}|^4} \quad .$$

We put $\Delta = 0$ in \mathbf{T} and perform the integration first over the angles of \mathbf{q} and then over its modulus. As a result, the main in $q_0/|\boldsymbol{\rho}|$ contribution is independent of q_0 and proportional to $Z\alpha$. Taking the sum of the contributions from these two regions and performing the integration over the energy ε , we get

$$\begin{aligned} M_{+--} &= \frac{4iN(\mathbf{e}\boldsymbol{\rho})^3}{\boldsymbol{\rho}^4} (\Delta \times \boldsymbol{\rho})_z \quad , \quad N = \frac{4Z\alpha e^3 \omega_2 \omega_3}{\pi \omega_1 \Delta^2 \boldsymbol{\rho}^2} ; \\ M_{+++} &= N \left[\mathbf{e}^* \Delta + 2(\mathbf{e}\Delta) \frac{(\mathbf{e}^* \boldsymbol{\rho})^2}{\boldsymbol{\rho}^2} \left(1 + \frac{\omega_2 - \omega_3}{\omega_1} \ln \frac{\omega_3}{\omega_2} + \right. \right. \\ &\quad \left. \left. \frac{\omega_2^2 + \omega_3^2}{2\omega_1^2} (\ln^2 \frac{\omega_3}{\omega_2} + \pi^2) \right) \right] \\ M_{++-} &= N \left[\mathbf{e}\Delta + 2(\mathbf{e}^* \Delta) \frac{(\mathbf{e}\boldsymbol{\rho})^2}{\boldsymbol{\rho}^2} \left(1 + \frac{\omega_1 + \omega_3}{\omega_2} (\ln \frac{\omega_3}{\omega_1} + i\pi) + \right. \right. \\ &\quad \left. \left. \frac{\omega_1^2 + \omega_3^2}{2\omega_2^2} (\ln^2 \frac{\omega_3}{\omega_1} + 2i\pi \ln \frac{\omega_3}{\omega_1}) \right) \right] \quad , \end{aligned} \quad (28)$$

where A_z is the projection of the vector \mathbf{A} on the direction of \mathbf{k}_1 . Substituting (28) into (27) and performing the elementary integration over the angles of vectors Δ and $\boldsymbol{\rho}$, we come to the expression

$$d\sigma = \frac{4Z^2 \alpha^5}{\pi^2} \frac{d\rho^2 d\Delta^2 dx}{\rho^4 \Delta^2} g(x) \quad , \quad (29)$$

where the function $g(x)$ for different polarizations has the form

$$g_{+--}(x) = x(1-x) \quad , \quad (30)$$

$$\begin{aligned}
g_{+++}(x) &= \frac{1}{2}x(1-x) \left[1 + \left(1 + (2x-1) \ln \left(\frac{1-x}{x} \right) + \right. \right. \\
&\quad \left. \left. \frac{x^2 + (1-x)^2}{2} \left(\ln^2 \left(\frac{1-x}{x} \right) + \pi^2 \right) \right)^2 \right] , \\
g_{++-}(x) &= \frac{1}{2}x(1-x) \left[1 + \left| 1 + (2/x-1)(\ln(1-x) + i\pi) + \right. \right. \\
&\quad \left. \left. \frac{1 + (1-x)^2}{2x^2} (\ln^2(1-x) + 2i\pi \ln(1-x)) \right|^2 \right] , \\
g_{+-+}(x) &= g_{++-}(1-x) .
\end{aligned}$$

Formulae (29) and (30) are in agreement with the corresponding results of [16], obtained in the Weizsäcker-Williams approximation. However, this approach does not allow to obtain the amplitudes (28) themselves. The large logarithm appears after the integration of (29) over Δ^2 from Δ_{min}^2 up to ρ^2 where $\Delta_{min} \sim r_c^{-1}$ for the screened Coulomb potential and $\Delta_{min} \sim \rho^2/\omega_1$ for the pure Coulomb case. It is interesting to compare the contributions of different helicity amplitudes to the cross section at $\Delta \rightarrow 0$. In Fig. 3 the function $g(x)$ is shown for different helicities as well as the quantity

$$\bar{g}(x) = g_{+--}(x) + g_{+++}(x) + g_{++-}(x) + g_{+-+}(x) , \quad (31)$$

which corresponds to the summation over the final photon polarizations. It is seen that $\bar{g}(x)$ has a wide plateau.

The Coulomb corrections to the photon-splitting amplitude at $\Delta \rightarrow 0$ are small compared to the Born

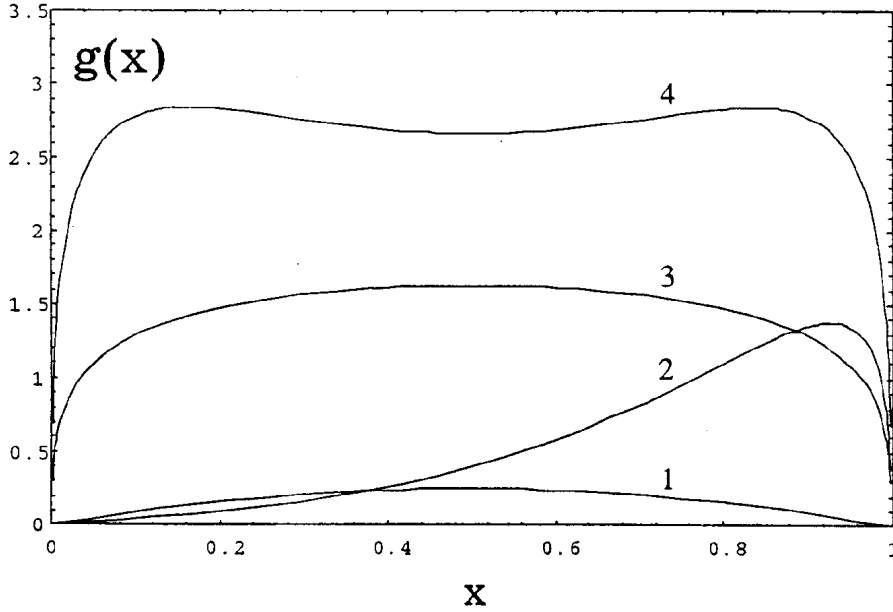


Figure 3: Function $g(x)$ from eq. (30) for different polarizations: $g_{+--}(x)$ (curve 1), $g_{++-}(x)$ (curve 2), $g_{+++}(x)$ (curve 3), and $\bar{g}(x)$ (curve 4), eq. (31).

term (28). One can show that in this case the Coulomb corrections are proportional to $(Z\alpha)^3 \Delta \ln^2(\Delta/\rho)$. Thus, they become essential at $\Delta \sim \rho$. We shall present the detailed investigation of the Coulomb corrections elsewhere.

8 Born approximation

As it was mentioned above, the photon-splitting amplitude obtained in the lowest Born approximation [14, 15] at arbitrary energies and momentum transfers is cumbersome and rather difficult to use. Therefore, it is interesting to consider the first term of $Z\alpha$ expansion of (24). To obtain this asymptotics we perform the substitution

$$(\mathbf{e}\nabla_{\mathbf{q}}) \operatorname{Im} \left(\frac{|\mathbf{q} + \mathbf{\Delta}|}{|\mathbf{q} - \mathbf{\Delta}|} \right)^{2iZ\alpha} \rightarrow Z\alpha \left[\frac{1}{(\mathbf{e}^*, \mathbf{q} + \mathbf{\Delta})} - \frac{1}{(\mathbf{e}^*, \mathbf{q} - \mathbf{\Delta})} \right]$$

in (24) and write the quantities \mathcal{D}_{1-3} in (25) as

$$\begin{aligned} \mathcal{D}_1 &= \left(\mathbf{q} + \frac{\kappa_2 - \kappa_3}{\omega_1} \mathbf{\Delta} \right)^2 - \frac{4\omega_2\omega_3\kappa_2\kappa_3}{\omega_1^2} \mathbf{f}_{23}^2 - i0 \quad , \\ \mathcal{D}_3 &= \left(\mathbf{q} + \frac{\kappa_2 - \epsilon}{\omega_2} \mathbf{\Delta} \right)^2 + \frac{4\omega_1\omega_3\kappa_2\epsilon}{\omega_2^2} \mathbf{f}_3^2 \quad , \\ \mathcal{D}_2 &= \left(\mathbf{q} - \frac{\kappa_3 + \epsilon}{\omega_3} \mathbf{\Delta} \right)^2 - \frac{4\omega_1\omega_2\kappa_3\epsilon}{\omega_3^2} \mathbf{f}_2^2 \quad . \end{aligned} \quad (32)$$

After that we shift the variable of integration \mathbf{q} in each term so that the quantities \mathcal{D}_{1-3} become independent of the angle ϕ of the vector \mathbf{q} . For instance, in the terms containing \mathcal{D}_1 we make a substitution $\mathbf{q} \rightarrow \mathbf{q} - \frac{\kappa_2 - \kappa_3}{\omega_1} \mathbf{\Delta}$. As a result, one can pass to the variable $z = \exp(i\phi)$ and take easily the contour integral. Taking the integrals with respect to $|\mathbf{q}|$ and ϵ , we get for the Born amplitudes

$$M_{+--} = \frac{2iZ\alpha e^3 (\mathbf{f}_2 \times \mathbf{f}_3)_z}{\pi \mathbf{\Delta}^2 (\mathbf{e}^* \mathbf{f}_2) (\mathbf{e}^* \mathbf{f}_3) (\mathbf{e}^* \mathbf{f}_{23})} \quad , \quad (33)$$

$$\begin{aligned} M_{+++} &= \frac{2(Z\alpha)e^3\omega_1}{\pi \mathbf{\Delta}^2 (\mathbf{e} \mathbf{f}_{23})^2 \omega_2 \omega_3} \left\{ (\mathbf{e} \mathbf{\Delta}) \left[1 + \frac{(\mathbf{e} \mathbf{f}_2) + (\mathbf{e} \mathbf{f}_3)}{(\mathbf{e} \mathbf{f}_{23})} \ln\left(\frac{a_2}{a_3}\right) + \right. \right. \\ &\left. \frac{(\mathbf{e} \mathbf{f}_2)^2 + (\mathbf{e} \mathbf{f}_3)^2}{(\mathbf{e} \mathbf{f}_{23})^2} \left(\frac{\pi^2}{6} + \frac{1}{2} \ln^2\left(\frac{a_2}{a_3}\right) + \operatorname{Li}_2(1 - a_2) + \operatorname{Li}_2(1 - a_3) \right) \right] + \\ &\frac{1}{(\mathbf{e} \mathbf{\Delta})} \left[\omega_3^2 (\mathbf{e} \mathbf{f}_3)^2 \frac{a_2}{1 - a_2} \left(1 + \frac{a_2 \ln(a_2)}{1 - a_2} \right) + \omega_2^2 (\mathbf{e} \mathbf{f}_2)^2 \frac{a_3}{1 - a_3} \times \right. \\ &\left. \left(1 + \frac{a_3 \ln(a_3)}{1 - a_3} \right) \right] + \frac{2(\mathbf{e} \mathbf{f}_2)(\mathbf{e} \mathbf{f}_3)}{(\mathbf{e} \mathbf{f}_{23})} \left[\omega_3 \frac{a_2 \ln(a_2)}{1 - a_2} - \omega_2 \frac{a_3 \ln(a_3)}{1 - a_3} \right] \right\} \quad , \end{aligned}$$

$$\begin{aligned} M_{++-} &= \frac{2(Z\alpha)e^3\omega_2}{\pi \mathbf{\Delta}^2 (\mathbf{e}^* \mathbf{f}_3)^2 \omega_1 \omega_3} \left\{ (\mathbf{e}^* \mathbf{\Delta}) \left[1 - \frac{(\mathbf{e}^* \mathbf{f}_2) + (\mathbf{e}^* \mathbf{f}_{23})}{(\mathbf{e}^* \mathbf{f}_3)} \ln\left(\frac{-a_1}{a_2}\right) + \right. \right. \\ &\left. \frac{(\mathbf{e}^* \mathbf{f}_2)^2 + (\mathbf{e}^* \mathbf{f}_{23})^2}{(\mathbf{e}^* \mathbf{f}_3)^2} \left(\frac{\pi^2}{6} + \frac{1}{2} \ln^2\left(\frac{-a_1}{a_2}\right) + \operatorname{Li}_2(1 - a_2) + \operatorname{Li}_2(1 + a_1) \right) \right] + \\ &\frac{1}{(\mathbf{e}^* \mathbf{\Delta})} \left[\omega_3^2 (\mathbf{e}^* \mathbf{f}_{23})^2 \frac{a_2}{1 - a_2} \left(1 + \frac{a_2 \ln(a_2)}{1 - a_2} \right) - \omega_1^2 (\mathbf{e}^* \mathbf{f}_2)^2 \frac{a_1}{1 + a_1} \times \right. \\ &\left. \left(1 - \frac{a_1 \ln(-a_1)}{1 + a_1} \right) \right] + \frac{2(\mathbf{e}^* \mathbf{f}_2)(\mathbf{e}^* \mathbf{f}_{23})}{(\mathbf{e}^* \mathbf{f}_3)} \left[\omega_1 \frac{a_1 \ln(-a_1)}{1 + a_1} - \omega_3 \frac{a_2 \ln(a_2)}{1 - a_2} \right] \right\} \quad , \end{aligned}$$

where

$$a_1 = \frac{\mathbf{\Delta}^2}{\omega_2 \omega_3 \mathbf{f}_{23}^2} \quad , \quad a_2 = \frac{\mathbf{\Delta}^2}{\omega_1 \omega_2 \mathbf{f}_2^2} \quad , \quad a_3 = \frac{\mathbf{\Delta}^2}{\omega_1 \omega_3 \mathbf{f}_3^2} \quad ,$$

$$\text{Li}_2(x) = - \int_0^x \frac{dt}{t} \ln(1-t) \quad .$$

It follows from (25) that $\ln(-a_1)$ should be interpreted as $\ln(-a_1 + i0) = \ln(a_1) + i\pi$. Besides,

$$\text{Li}_2(1 + a_1) = \text{Li}_2(1 + a_1 - i0) = \frac{\pi^2}{6} - \ln(1 + a_1)[\ln(a_1) + i\pi] - \text{Li}_2(-a_1)$$

The result (33) is obtained for $|\mathbf{\Delta}_\perp| \gg |\mathbf{\Delta}_\parallel|$. One can show that it remains valid in the case $|\mathbf{\Delta}_\perp| \sim |\mathbf{\Delta}_\parallel|$ if the expression (5) for $\mathbf{\Delta}^2$ is used in (33). Actually, in eq. (33) the difference between $\mathbf{\Delta}^2$ and $\mathbf{\Delta}_\perp^2$ is essential only in the overall factor $1/\mathbf{\Delta}^2$. For a screened Coulomb potential the amplitudes (33) should be multiplied by the atomic form factor $(1 - F(\mathbf{\Delta}^2))$. For the case of Molière potential [19] it reads

$$1 - F(\mathbf{\Delta}^2) = \Delta^2 \sum_{i=1}^3 \frac{\alpha_i}{\Delta^2 + \beta_i^2}, \quad (34)$$

where

$$\alpha_1 = 0.1 \quad , \quad \alpha_2 = 0.55 \quad , \quad \alpha_3 = 0.35 \quad , \quad \beta_i = \beta_0 b_i \quad , \quad (35)$$

$$b_1 = 6 \quad , \quad b_2 = 1.2 \quad , \quad b_3 = 0.3 \quad , \quad \beta_0 = mZ^{1/3}/121 \quad .$$

Remind that the representation (33) is valid when $|\mathbf{k}_{2\perp}|, |\mathbf{k}_{3\perp}| \gg m$.

9 Cross section

As it was suggested in [13], to overcome the problems of background in the measurement of photon splitting one has to register the events with $|\mathbf{f}_{2,3}| \geq f_0$ where $f_0 \ll 1$ is the angle determined by the experimental conditions. Let us consider the cross section integrated over \mathbf{f}_3 for $|\mathbf{f}_3| > f_0$. It is interesting to compare the exact result for this cross section ($d\sigma/dx d\mathbf{f}_2$) following from (33) and (26) with that obtained in the Weizsäcker-Williams approximation ($d\sigma_{approx}/dx d\mathbf{f}_2$). The large logarithm corresponds to the contribution of the region $\Delta \ll \rho = |\omega_2 \mathbf{f}_2 - \omega_3 \mathbf{f}_3|/2$ where $f_3 \approx x f_2/(1-x)$. Taking the integral over Δ^2 in eq. (29) from Δ_{min}^2 up to Δ_{eff}^2 , where (see [16])

$$\Delta_{min}^2 = \Delta_\parallel^2 = (\omega_1 f_2^2 x/2(1-x))^2 \quad , \quad \Delta_{eff}^2 = \rho^2 = (\omega_1 x f_2)^2 \quad ,$$

and summing over the final photon polarizations we get for a pure Coulomb potential

$$\frac{d\sigma_{approx}}{dx d\mathbf{f}_2} = \frac{8Z^2 \alpha^5}{\pi^3 \omega_1^2} \frac{\bar{g}(x)}{x^2 f_2^4} \ln\left(\frac{2(1-x)}{f_2}\right) \vartheta\left(\frac{x}{1-x} f_2 - f_0\right) \quad . \quad (36)$$

For the case of a screened Coulomb potential the approximate cross section is

$$\frac{d\sigma_{approx}}{dx d\mathbf{f}_2} = \frac{4Z^2 \alpha^5}{\pi^3 \omega_1^2} \frac{\bar{g}(x)}{x^2 f_2^4} \left[2 \ln\left(\frac{\omega_1 x f_2}{\beta_0}\right) + \gamma \right] \vartheta\left(\frac{x}{1-x} f_2 - f_0\right) \quad . \quad (37)$$

The function γ in eq. (37) is

$$\gamma = 1 - \sum_{i=1}^3 \alpha_i^2 (\ln a_i + 1) - 2 \sum_{i>j} \alpha_i \alpha_j \frac{a_i \ln a_i - a_j \ln a_j}{a_i - a_j} \quad , \quad a_i = b_i^2 + \Delta_{min}^2/\beta_0^2 \quad (38)$$

and the coefficients α_i , b_i and β_0 are defined in (35). If $\Delta_{min}^2/\beta_0^2 \gg 1$ then $\gamma = -\ln(\Delta_{min}^2/\beta_0^2)$ and eq. (37) turns to eq. (36). If $\Delta_{min}^2/\beta_0^2 \ll 1$ then $\gamma = -0.158$. For the case of a pure Coulomb potential the dependence of $\sigma_0^{-1} d\sigma/dx d\mathbf{f}_2$ on f_2/f_0 is shown in Fig. 4 at $f_0 = 10^{-3}$ and $x = 0.7$ (curve 1), $x = 0.3$ (curve 2), where

$$\sigma_0 = \frac{4Z^2 \alpha^5 \bar{g}(x)}{\pi^3 \omega_1^2 f_0^4} \quad ,$$

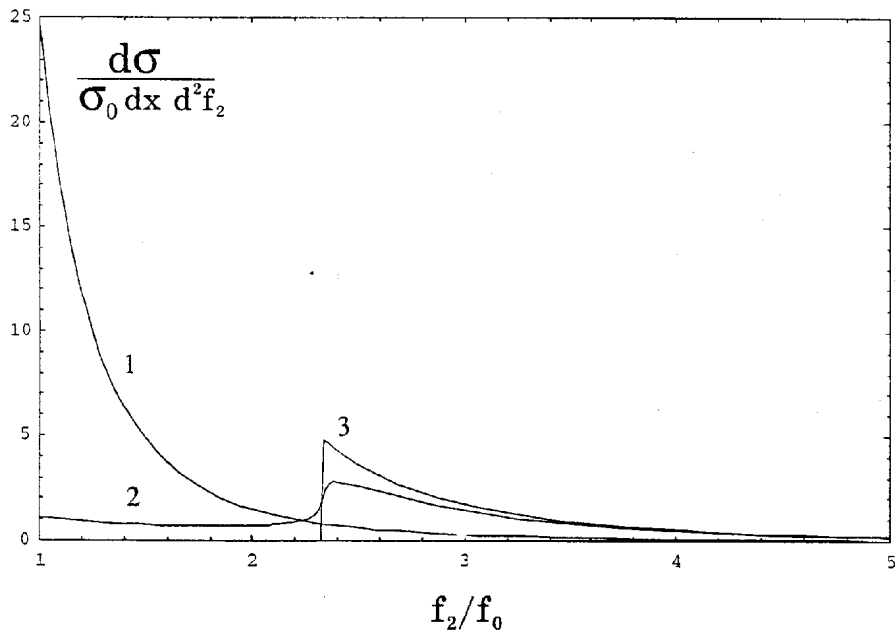


Figure 4: $\sigma_0^{-1} d\sigma/dx d^2\mathbf{f}_2$ versus f_2/f_0 for the case of a pure Coulomb potential, $f_0 = 10^{-3}$, $x = 0.7$ (1), $x = 0.3$ (2), σ_0 is given in the text. Curve 3 corresponds to the Weizsäcker-Williams approximation for $x = 0.3$.

and $\bar{g}(x)$ is defined in (31). For $x = 0.7$ the cross section given by (36) practically coincides with the curve 1. For $x = 0.3$ it is not the case (the curve 3 corresponds to the cross section in the Weizsäcker-Williams approximation at $x = 0.3$). However, within a good accuracy the cross section $d\sigma/dx$ at $x = 0.3$ agrees with that obtained from eq. (36). It should be so, since $d\sigma/dx$ is invariant with respect to the substitution $x \rightarrow 1-x$ and at $x = 0.7$, as we pointed out above, the approximate result (36) is in accordance with the exact one. At $x = 0.5$ the difference between the exact result and the approximate one is rather essential (see Fig. 5). It can be explained as follows. The large logarithm appears as a result of integration with respect to \mathbf{f}_3 over the range $|(1-x)\mathbf{f}_3 + x\mathbf{f}_2| \ll x\mathbf{f}_2$. After the integration over the azimuth angle φ between vectors \mathbf{f}_2 and $-\mathbf{f}_3$ we should integrate over f_3 from f_0 up to $x f_2/(1-x)$ and from $x f_2/(1-x)$ to infinity. If $x f_2/(1-x) \approx f_0$ then the contribution of the first region vanishes and the cross section becomes approximately two times smaller (in accordance with Fig. 5).

Since the amplitudes (33) are obtained in zero-mass limit, it is interesting to estimate the accuracy of this approximation. Numerical calculations of the Born cross section $d\sigma/dx d\mathbf{f}_2$ with the electron mass taken into account were performed in [17] (see Table V of that paper) for $Z = 79$, $x = 0.87$, $\omega_1 = 1.7$ GeV, 3.4 GeV, 6.1 GeV and for five values of the angle f_2 in the interval (1.2 – 2.8) mrad. These results are compared with ours in Table. One can see a good agreement everywhere. Only in one point corresponding to the smallest value of the transverse momentum $k_{2\perp} = 1.77$ MeV the accuracy is 7%.

Table. The photon-splitting cross section $d\sigma/\omega_1 dx d\mathbf{f}_2$ in b/GeV sr for $Z = 79$, $x = 0.87$

f_2 (mrad)	$\omega_1 = 1.7$ GeV		$\omega_1 = 3.4$ GeV		$\omega_1 = 6.1$ GeV	
	Present result	Paper [17]	Present result	Paper [17]	Present result	Paper [17]
1.2	22.72	21.1	3.11	3.25	0.565	0.56
1.6	7.47	7.4	0.994	1.03	0.177	0.18
2.0	3.09	3.12	0.404	0.41	0.0712	0.072
2.4	1.48	1.51	0.191	0.19	0.0335	0.034
2.8	0.793	0.80	0.101	0.10	0.0177	0.018

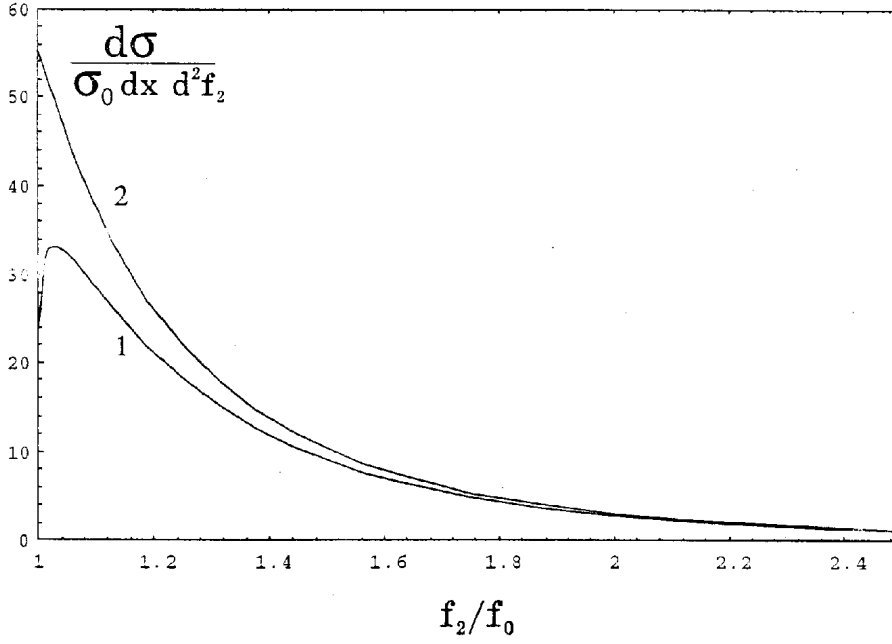


Figure 5: Same as Fig. 4 but for $x = 0.5$ (1). Curve 2 corresponds to the Weizsäcker-Williams approximation.

For the case of a pure Coulomb potential the cross section $d\sigma/dx$ can be approximated with a good accuracy by the following formula

$$\frac{d\sigma_{coul}}{dx} = \pi f_0^2 \sigma_0 \left[\frac{\vartheta(x-1/2)}{x^2} \left(2 \ln \frac{2(1-x)}{f_0} - 1 - F(x) \right) + (x \leftrightarrow 1-x) \right], \quad (39)$$

where

$$F(x) = \frac{1}{2} + \frac{x}{1-x} + \frac{2x-1}{(1-x)^2} \ln \left(2 - \frac{1}{x} \right). \quad (40)$$

If $\omega_1^2 f_0^4 / \beta_0^2 \ll 1$ then the corresponding expression for the cross section in a screened Coulomb potential reads

$$\frac{d\sigma_{scr}}{dx} = \pi f_0^2 \sigma_0 \left[\frac{\vartheta(x-1/2)}{x^2} \left(2 \ln \frac{\omega_1 x f_0}{\beta_0} + 0.842 - F(x) \right) + (x \leftrightarrow 1-x) \right], \quad (41)$$

with $F(x)$ defined in (40). The function $F(x)$ characterizes the difference between the exact cross section and that obtained in the Weizsäcker-Williams approximation. It is seen from Fig. 6 that this difference can be significant only at x being close to 0.5. As for the total cross section, the difference between the exact result and the approximate one is about a few per cent.

The inequality $\Delta \ll \rho$ which provides the applicability of the Weizsäcker-Williams approximation corresponds to a small angle φ between the vectors \mathbf{f}_2 and $-\mathbf{f}_3$ (when the vectors \mathbf{f}_2 and \mathbf{f}_3 have almost opposite directions). So, it is interesting to consider the quantity $d\sigma(\varphi_{max})/dx$ which is the cross section integrated over the angle φ from $-\varphi_{max}$ to φ_{max} . In the case of a pure Coulomb potential the dependence of $(\pi f_0^2 \sigma_0)^{-1} d\sigma(\varphi_{max})/dx$ on φ_{max} is shown in Fig. 7 for different x and $f_0 = 10^{-3}$. One can see that the cross section becomes close to its total value at relatively large φ_{max} . The same conclusion is also valid for the case of a screened Coulomb potential.

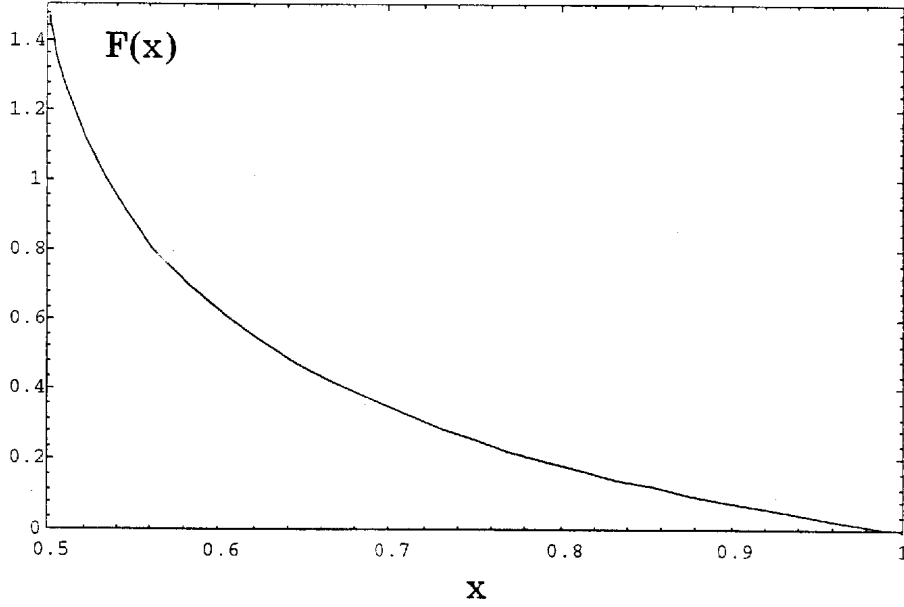


Figure 6: Function $F(x)$, eq. (40).

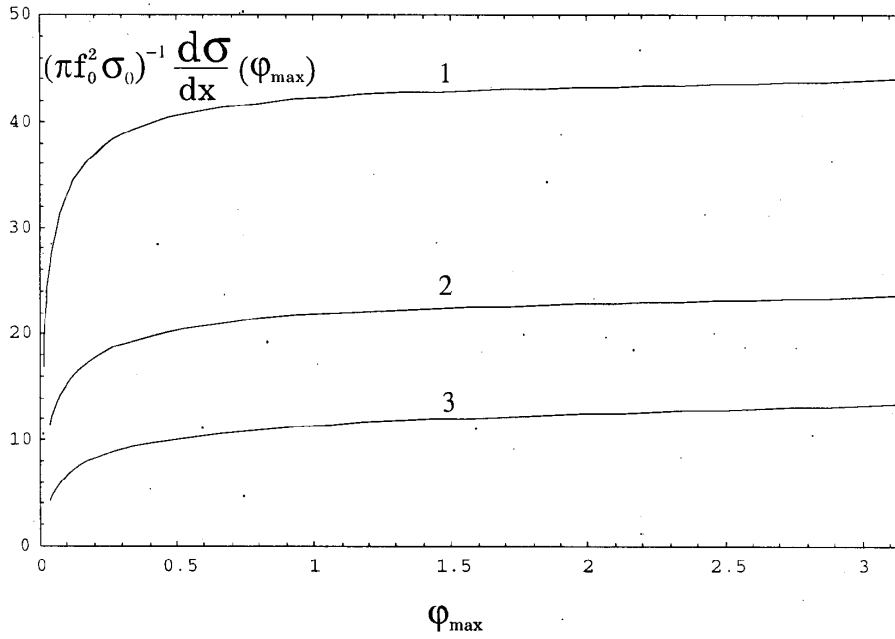


Figure 7: The dependence of $(\pi f_0^2 \sigma_0)^{-1} d\sigma(\varphi_{max})/dx$ on φ_{max} for a pure Coulomb case at different x : $x = 0.5$ (1), $x = 0.7$ (2), and $x = 0.9$ (3); $f_0 = 10^{-3}$.

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