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THE THEORY OF LANDAU, POMERANCHUK, MIGDAL EFFECT

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Abstract

Bremsstrahlung of photons from highly relativistic electrons is investigated. The cross section of the processes, which is suppressed due to a multiple scattering of an emitting electron in dense media (LPM effect) and due to photon interaction with electrons of a medium, is calculated with an accuracy up to "next to leading logarithm" and with the Coulomb corrections taken into account. Making allowances for a multiple scattering and a polarization of a medium an analysis of radiation on a target boundary is carried out. The method of consideration of radiation in a thin target under influence of the LPM effect is developed. Interrelation with the recent experiment is discussed.

1 Introduction

When a high-energy electron emits a soft photon via bremsstrahlung, the process occurs over a rather long distance, known as the formation length. If anything happens to an electron or a photon while traveling this distance, the emission can be disrupted. Landau and Pomeranchuk were the first who showed that if the formation length of bremsstrahlung becomes comparable to the distance over which a multiple scattering becomes important, the bremsstrahlung will be suppressed [1]. Migdal [2], [3] developed a quantitative theory of this phenomenon. Side by side with the multiple scattering of emitting electron one has to take into account also an influence of a medium on radiated electromagnetic field. Since long distances are essential in the problem under consideration this can be done by introducing dielectric constant $\varepsilon(\omega)$. This effect leads also to suppression of the soft photon emission (Ter-Mikaelian effect, see in [4]). A clear qualitative analysis of different mechanisms of suppression is presented in [5],[6]. More simple derivation of the Migdal's results is given in [7].

The next step in a quantitative theory of LPM effect was made in [8]. This theory is based on the quasiclassical operator method in QED developed by authors [7], [9]. One of the basic equations (obtained with use of kinetic equations describing a motion of electron in a medium in the presence of external field) is the Schrödinger equation in external field with imaginary potential (Eq.(3.3),[8]). The same equation (without external field) was rederived recently in [10]. The last derivation is based on the approach results of which is coincide basically with the operator quasiclassical method. In [11] a new calculation approach is developed where multiple scattering is described with the path integral treatment.

New activity with the theory of LPM effect is connected with a very successful series of experiments [12] - [14] performed at SLAC during last years (see in this connection [15]). In these experiments the cross section of bremsstrahlung of soft photons with energy from 200 KeV to 500 MeV from electrons with energy 8 GeV and 25 GeV is measured with an accuracy of the order of a few percent. Both LPM and dielectric suppression is observed and investigated. These experiments are the challenge for the theory since in all the mentioned papers calculations are performed to logarithmic accuracy which is not enough for description of the new experiment. The contribution of the Coulomb corrections (at least for heavy elements) is larger then experimental errors and these corrections should be taken into account.

In the present paper we calculated the cross section of bremsstrahlung process with term $\propto 1/L$, where L is characteristic logarithm of the problem, and with the Coulomb corrections taken into account (Section 2 and Appendix A). This cross section is valid for very high energies when the LPM effect manifest itself for a photon energy of the order of an energy of the initial electron. In the photon energy region, where the LPM effect is "turned off", our cross section gives the exact Bethe-Heitler cross section (within power accuracy) with Coulomb corrections. This important feature was absent in the previous calculations. The polarization of a medium is incorporated into this approach (Section 3). The considerable contribution into the soft part of the investigated in the experiment spectrum of radiation gives a photon emission on the boundaries of a target. We calculated this contribution taking into account the multiple scattering and polarization of a medium for the case when a target is much thicker than the formation length of the radiation (Section 4). In Section 5 we considered a case when a target is much thinner than the formation length. In this case the cross section has multiplicative form (probability of radiation times cross section of scattering for the given impact parameter). In Section 6 a case of an intermediate thickness of a target (between cases of a thick and a thin target) is analyzed, polarization of a medium is not included. In Section 7 a qualitative picture of a spectral curve (an effective thickness of a target, position of a minimum) is discussed. In Section 8 we compare the theoretical curve for the intensity spectrum with the data. Although agreement between experiment and theory is rather satisfactory, an additional analysis should be done to obtain information about an accuracy of agreement between experimental data and theory.

2 The LPM effect in an infinitely thick target

As well known (see, e.g. [16], Sec.93) the formation length of radiation is (in this paper the system $\hbar = c = 1$ is used)

$$l_c = \frac{2\varepsilon\varepsilon'}{m^2\omega\zeta}, \quad \zeta = 1 + \gamma^2\vartheta^2, \quad \varepsilon' = \varepsilon - \omega, \quad \gamma = \frac{\varepsilon}{m}, \tag{2.1}$$

where ε is the energy of the initial electron, ω is the energy of radiated photon, ϑ is the angle between momenta of the photon and the initial electron. We consider first the case when the formation length is much shorter than thickness of a target $l (l_c \ll l)$. In this case the spectral distribution of the probability of radiation per unit time is given by expression (2.18), [8] (see also [9], Section 7.4)

$$\frac{dW}{d\omega} = \alpha\omega \mathrm{Re} \int\limits_0^\infty d\tau \exp(-i\frac{a\tau}{2}) \left[\frac{\omega^2}{\gamma^2\varepsilon'^2}\varphi_0(0,\tau) - i\left(1 + \frac{\varepsilon^2}{\varepsilon'^2}\right)\boldsymbol{\nabla}\boldsymbol{\varphi}(0,\tau)\right], \tag{2.2}$$

where $\alpha = e^2 = \dfrac{1}{137}$, functions $\varphi_\mu(\varphi_0, \boldsymbol{\varphi})$ satisfy an equation

$$\frac{\partial\varphi_\mu}{\partial\tau} - \frac{ib}{2}\Delta\varphi_\mu(\mathbf{x},\tau) = n(\Sigma(\mathbf{x}) - \Sigma(0))\varphi_\mu(\mathbf{x},\tau) \tag{2.3}$$

with the initial conditions

$$\varphi_0(\mathbf{x},0) = \delta(\mathbf{x}), \quad \boldsymbol{\varphi}(\mathbf{x},0) = -i\boldsymbol{\nabla}\delta(\mathbf{x}). \tag{2.4}$$

Here n is the number density of atoms in the medium, $\mathbf{x}$ is the coordinate in two-dimensional space conjugated to the space (two-dimensional) of radiation angle $\boldsymbol{\vartheta}$, $\Sigma(\mathbf{x})$ is the Fourier transform of the scattering cross section:

$$\Sigma(\mathbf{x}) = \int d^2\vartheta \exp(i\mathbf{x}\boldsymbol{\vartheta})\sigma(\boldsymbol{\vartheta}), \quad a = \frac{\omega m^2}{\varepsilon\varepsilon'}, \quad b = \frac{\omega\varepsilon}{\varepsilon'}. \tag{2.5}$$

For a screened Coulomb potential we have

$$\sigma(\vartheta) = \frac{4Z^2\alpha^2}{\varepsilon^2(\vartheta^2 + \vartheta_1^2)^2}, \quad \Sigma(x) = 4\pi\frac{Z^2\alpha^2}{\varepsilon^2}\frac{x}{\vartheta_1}K_1(x\vartheta_1), \tag{2.6}$$

where $\vartheta_1 = \dfrac{1}{a_s\varepsilon}$, a_s is the screening radius ($a_s = 0.81 a_B Z^{-1/3}$, a_B is the Bohr radius), K_1 is the modified Bessel function. As we will show below, the main contribution to the probability is given by

$$\frac{1}{x} \sim \vartheta \geq \frac{1}{\gamma} \gg \vartheta_1 = \frac{\lambda_c}{a_s\gamma},$$

where $\lambda_c = \dfrac{1}{m} = \left(\dfrac{\hbar}{mc}\right)$ is the electron Compton wavelength. Expanding $K_1(x\vartheta_1)$ as a power series in $x\vartheta_1$ and introducing new variables

$$t = \frac{a}{2}\tau, \quad \boldsymbol{\varrho} = \sqrt{\frac{a}{b}}\mathbf{x} = \frac{1}{\gamma}\mathbf{x}, \tag{2.7}$$

we obtain for the spectral distribution of the probability of radiation

$$\frac{dW}{d\omega} = \frac{2\alpha}{\gamma^2}\mathrm{Re}\int\limits_0^{\infty} dt e^{-it}\left[R_1\varphi_0(0,t) + R_2\mathbf{p}\boldsymbol{\varphi}(0,t)\right], \tag{2.8}$$

where $R_1 = \dfrac{\omega^2}{\varepsilon\varepsilon'}$, $\quad R_2 = \dfrac{\varepsilon}{\varepsilon'} + \dfrac{\varepsilon'}{\varepsilon}$, and the functions φ_μ now satisfy an equation

$$\begin{aligned} &i\frac{\partial\varphi_\mu}{\partial t} = \left(\mathbf{p}^2 - iV(\boldsymbol{\varrho})\right)\varphi_\mu, \quad \mathbf{p} = -i\boldsymbol{\nabla}_{\boldsymbol{\varrho}}, \quad V(\boldsymbol{\varrho}) = -Q\boldsymbol{\varrho}^2\Big(\ln\gamma^2\vartheta_1^2 \\ &+\ln\frac{\boldsymbol{\varrho}^2}{4} + 2C - 1\Big), \quad Q = \frac{2\pi nZ^2\alpha^2\varepsilon\varepsilon'}{m^4\omega}, \quad C = 0.577... \end{aligned} \tag{2.9}$$

with the initial conditions $\varphi_0(\boldsymbol{\varrho},0) = \delta(\boldsymbol{\varrho})$, $\quad \boldsymbol{\varphi}(\boldsymbol{\varrho},0) = \mathbf{p}\delta(\boldsymbol{\varrho})$, the functions φ_0 and $\boldsymbol{\varphi}$ in (2.8) are rescaled according with the initial conditions (factors $1/\gamma^2$ and $1/\gamma^3$, correspondingly). Note, that it is implied that in formulae (2.2),(2.8) subtraction at $V = 0$ is made.

The potential $V(\boldsymbol{\varrho})$ (2.9) corresponds to consideration of scattering in the Born approximation. The difference of exact as a function of $Z\alpha$ potential $V(\boldsymbol{\varrho})$ and taken in the Born approximation is computed in Appendix A. The potential $V(\boldsymbol{\varrho})$ with the Coulomb corrections taken into account is

$$\begin{aligned} V(\boldsymbol{\varrho}) &= -Q\boldsymbol{\varrho}^2\Big(\ln\gamma^2\vartheta_1^2 + \ln\frac{\boldsymbol{\varrho}^2}{4} + 2C - 1 + 2f(Z\alpha)\Big) \\ &= -Q\boldsymbol{\varrho}^2\Big(\ln\gamma^2\vartheta_2^2 + \ln\frac{\boldsymbol{\varrho}^2}{4} + 2C\Big), \end{aligned} \tag{2.10}$$

where $\vartheta_2 = \vartheta_1\exp(f - 1/2)$, the function $f = f(Z\alpha)$ see in (A.10).

In above formulae $\boldsymbol{\varrho}$ is space of the impact parameters measured in the Compton wavelengths λ_c, which is conjugate to space of the transverse momentum transfers measured in the electron mass m. An operator form of a solution of Eq. (2.9) is

$$\begin{aligned} &\varphi_0(\boldsymbol{\varrho},t) = \exp(-iHt)\varphi_0(\boldsymbol{\varrho},0) =<\boldsymbol{\varrho}|\exp(-iHt)|0>, \quad H = \mathbf{p}^2 - iV(\boldsymbol{\varrho}), \\ &\boldsymbol{\varphi}(\boldsymbol{\varrho},t) = \exp(-iHt)\mathbf{p}\varphi_0(\boldsymbol{\varrho},0) =<\boldsymbol{\varrho}|\exp(-iHt)\mathbf{p}|0>, \end{aligned} \tag{2.11}$$

where we introduce the Dirac state vectors: $|\boldsymbol{\varrho}>$ is the state vector of coordinate $\boldsymbol{\varrho}$, $<\boldsymbol{\varrho}|0>=\delta(\boldsymbol{\varrho})$. Substituting (2.11) into (2.8) and taking integral over t we obtain for the spectral distribution of the probability of radiation

$$\frac{dW}{d\omega} = \frac{2\alpha}{\gamma^2}\mathrm{Im} < 0|R_1\left(G^{-1} - G_0^{-1}\right) + R_2\mathbf{p}\left(G^{-1} - G_0^{-1}\right)\mathbf{p}|0>, \tag{2.12}$$

where

$$G = \mathbf{p}^2 + 1 - iV, \quad G_0 = \mathbf{p}^2 + 1. \tag{2.13}$$

Here and below we consider an expression $< 0|...|0 >$ as a limit: $\lim \mathbf{x} \to 0$, $\lim \mathbf{x}' \to 0$ of $< \mathbf{x}|...|\mathbf{x}' >$.

Now we estimate effective impact parameters ϱ_c which give the main contribution into radiation probability. Since characteristic values of ϱ_c will be found straightforwardly at calculation of (2.12), we estimate characteristic angles ϑ_c connected with ϱ_c by an equality $\varrho_c = 1/(\gamma\vartheta_c)$. The mean square scattering angle of a particle on the formation length of a photon l_c (2.1) has the form

$$\vartheta_s^2 = \frac{4\pi Z^2\alpha^2}{\varepsilon^2}nl_c\ln\frac{\zeta}{\gamma^2\vartheta_1^2} = \frac{4Q}{\gamma^2\zeta}\ln\frac{\zeta}{\gamma^2\vartheta_1^2}. \tag{2.14}$$

When $\vartheta_s^2 \ll 1/\gamma^2$ the contribution in the probability of radiation gives a region where $\zeta \sim 1(\vartheta_c = 1/\gamma)$, in this case $\varrho_c = 1$. When $\vartheta_s \gg 1/\gamma$ the characteristic angle of radiation is determined by self-consistency arguments:

$$\vartheta_s^2 \simeq \vartheta_c^2 \simeq \frac{\zeta_c}{\gamma^2} = \frac{4Q}{\zeta_c\gamma^2}\ln\frac{\zeta_c}{\gamma^2\vartheta_1^2}, \quad \frac{4Q}{\zeta_c^2}\ln\frac{\zeta_c}{\gamma^2\vartheta_1^2} = 1, \quad 4Q\varrho_c^4\ln\frac{1}{\gamma^2\vartheta_1^2\varrho_c^2} = 1. \tag{2.15}$$

It should be noted that when characteristic impact parameter ϱ_c becomes smaller than a radius of nucleus R_n, the potential $V(\boldsymbol{\varrho})$ acquires an oscillator form (see Appendix B, Eq.(B.3))

$$V(\boldsymbol{\varrho}) = Q\boldsymbol{\varrho}^2\left(\ln\frac{a_s^2}{R_n^2} - 0.041\right). \tag{2.16}$$

Allowing for estimates (2.15) we present the potential $V(\boldsymbol{\varrho})$ (2.9) in the following form

$$\begin{aligned} &V(\boldsymbol{\varrho}) = V_c(\boldsymbol{\varrho}) + v(\boldsymbol{\varrho}), \quad V_c(\boldsymbol{\varrho}) = q\boldsymbol{\varrho}^2, \quad q = QL, \quad L = \ln\frac{1}{\gamma^2\vartheta_2^2\varrho_c^2}, \\ &v(\boldsymbol{\varrho}) = -\frac{q\boldsymbol{\varrho}^2}{L}\left(2C + \ln\frac{\boldsymbol{\varrho}^2}{4\varrho_c^2}\right). \end{aligned} \tag{2.17}$$

The inclusion of the Coulomb corrections $f(Z\alpha)$ and -1 into $\ln\vartheta_2^2$ diminishes effectively the correction $v(\boldsymbol{\varrho})$ to the potential $V_c(\boldsymbol{\varrho})$. In accordance with such division of the potential we present propagators in expression (2.12) as

$$G^{-1} - G_0^{-1} = G^{-1} - G_c^{-1} + G_c^{-1} - G_0^{-1} \tag{2.18}$$

where

$$G_c = \mathbf{p}^2 + 1 - iV_c, \quad G = \mathbf{p}^2 + 1 - iV_c - iv$$

This representation of the propagator G^{-1} permits one to expand it over "perturbation" v. Indeed, with an increase of q the relative value of the perturbation is diminished ($\frac{v}{V_c} \sim \frac{1}{L}$) since effective impact parameters diminish and, correspondingly, the value of logarithm L in (2.17) increases. The maximal value of L is determined by a size of a nucleus R_n

$$L_{max} = \ln\frac{a_{s2}^2}{R_n^2} \simeq 2\ln\frac{a_{s2}^2}{\lambda_c^2} \equiv 2L_1, \tag{2.19}$$

where $a_{s2} = a_s\exp(-f + 1/2)$. So, one can to redefine the parameters a_s and ϑ_1 to include the Coulomb corrections.

The matrix elements of the operator G_c^{-1} could be calculated explicitly. The exponential parametrization of the propagator is

$$G_c^{-1} = i\int\limits_0^\infty dt e^{-it}\exp(-iH_ct), \quad H_c = \mathbf{p}^2 - iq\boldsymbol{\varrho}^2 \tag{2.20}$$

Below we will use matrix elements of the operator $\exp(-iH_ct)$

$$< \boldsymbol{\varrho}_1|\exp(-iH_ct)|\boldsymbol{\varrho}_2 > \equiv K_c(\boldsymbol{\varrho}_1, \boldsymbol{\varrho}_2, t). \tag{2.21}$$

The function $K_c(\boldsymbol{\varrho}_1, \boldsymbol{\varrho}_2, t)$ satisfies the Schrödinger equation (2.9) over each of two (symmetrical) variables $\boldsymbol{\varrho}_1$ and $\boldsymbol{\varrho}_2$ with $V = q\boldsymbol{\varrho}^2$ and the initial condition

$$K_c(\boldsymbol{\varrho}_1, \boldsymbol{\varrho}_2, 0) = \delta(\boldsymbol{\varrho}_2 - \boldsymbol{\varrho}_1). \tag{2.22}$$

We will seek a solution in the form (see also [8])

$$K_c(\boldsymbol{\varrho}_1, \boldsymbol{\varrho}_2, t) = \exp\left[\alpha(t)(\boldsymbol{\varrho}_1^2 + \boldsymbol{\varrho}_2^2) + 2\beta(t)\boldsymbol{\varrho}_1\boldsymbol{\varrho}_2 + \gamma(t)\right].$$

Substituting this expression into (2.9) we find a set of equations for α, β, γ

$$\dot\alpha = 4i\alpha^2 - q, \quad \dot\beta = 4i\alpha\beta, \quad \dot\gamma = 4i\alpha. \tag{2.23}$$

The initial conditions for this set follows from definition (2.21):

$$\begin{aligned} &\lim_{t\to 0} < \boldsymbol{\varrho}_1|\exp(-iH_ct)|\boldsymbol{\varrho}_2 > \to < \boldsymbol{\varrho}_1|\exp(-iH_0t)|\boldsymbol{\varrho}_2 > = \\ &\frac{1}{(2\pi)^2}\int d^2p\exp\left(i(\boldsymbol{\varrho}_2 - \boldsymbol{\varrho}_1)\mathbf{p} - i\mathbf{p}^2t\right) = \frac{1}{4\pi it}\exp\left(\frac{i\left(\boldsymbol{\varrho}_2 - \boldsymbol{\varrho}_1\right)^2}{4t}\right) \\ &\equiv K_0(\boldsymbol{\varrho}_2, \boldsymbol{\varrho}_1, t), \end{aligned} \tag{2.24}$$

where $H_0 = \mathbf{p}^2$. From (2.24) one has the initial conditions at $t \to 0$

$$\alpha(t) \to \frac{i}{4t}, \quad \beta(t) \to -\frac{i}{4\pi}, \quad \gamma(t) \to -\ln(4\pi i t). \tag{2.25}$$

The solution of the set (2.23) satisfying these initial conditions is

$$\alpha(t) = \frac{i\nu}{4}\coth\nu t, \quad \beta(t) = -\frac{i\nu}{4\sinh\nu t}, \quad \gamma(t) = -\ln(\sinh\nu t) + \ln\frac{\nu}{4\pi i}, \tag{2.26}$$

where $\nu = 2\sqrt{iq}$. As a result, we obtain the following expression for the sought function

$$K_c(\boldsymbol{\varrho}_1, \boldsymbol{\varrho}_2, t) = \frac{\nu}{4\pi i\sinh\nu t}\exp\left\{\frac{i\nu}{4}\left[(\boldsymbol{\varrho}_1^2 + \boldsymbol{\varrho}_2^2)\coth\nu t - \frac{2}{\sinh\nu t}\boldsymbol{\varrho}_1\boldsymbol{\varrho}_2\right]\right\}. \tag{2.27}$$

Substituting formulae (2.20) and (2.27) in the expression for the spectral distribution of the probability of radiation (2.12) we have

$$\begin{aligned} \frac{dW_c}{d\omega} &= \frac{\alpha}{2\pi\gamma^2}\mathrm{Im}\,\Phi(\nu), \\ \Phi(\nu) &= \nu\int\limits_0^\infty dt e^{-it}\left[R_1\left(\frac{1}{\sinh z} - \frac{1}{z}\right) - i\nu R_2\left(\frac{1}{\sinh^2 z} - \frac{1}{z^2}\right)\right], \end{aligned} \tag{2.28}$$

where $z = \nu t$. This formula gives the spectral distribution of the probability of radiation derived by Migdal [2]. However, here Coulomb corrections are included into parameter ν in contrast to [2].

We now expand the expression $G^{-1} - G_c^{-1}$ over powers of v

$$G^{-1} - G_c^{-1} = G_c^{-1}(-iv)G_c^{-1} + G_c^{-1}(-iv)G_c^{-1}(-iv)G_c^{-1} + ... \tag{2.29}$$

Substituting this expansion in (2.18) and then in (2.12) we obtain decomposition of the probability of radiation.Let us note that for $Q \ll 1$ the sum of the probability of radiation $\frac{dW_c}{d\omega}$ (2.28) and the first term of the expansion (2.29) gives the Bethe-Heitler spectrum of radiation, see below (2.40). At $Q \geq 1$ the expansion (2.29) is a series over powers of $\frac{1}{L}$. It is important that variation of the parameter ϱ_c by a factor order of 1 has an influence on the dropped terms in (2.29) only.

In accordance with (2.18) and (2.29) we present the probability of radiation in the form

$$\frac{dW}{d\omega} = \frac{dW_c}{d\omega} + \frac{dW_1}{d\omega} + \frac{dW_2}{d\omega} + ... \tag{2.30}$$

The probability of radiation $\frac{dW_c}{d\omega}$ is defined by Eq.(2.28). In formula (2.12) with allowance for (2.18) there is expression

$$\begin{aligned} -i < 0|G^{-1} - G_c^{-1}|0 >= &\int\limits_0^\infty dt_1\int\limits_0^\infty dt_2 e^{-i(t_1+t_2)}\int d^2\varrho K_c(0, \boldsymbol{\varrho}, t_1)v(\boldsymbol{\varrho}) \\ \times K_c(\boldsymbol{\varrho}, 0, t_2) + &\int\limits_0^\infty dt_1\int\limits_0^\infty dt_2\int\limits_0^\infty dt_3 e^{-i(t_1+t_2+t_3)}\int d^2\varrho_1\int d^2\varrho_2 K_c(0, \boldsymbol{\varrho}_1, t_1) \\ \times v(\boldsymbol{\varrho}_1)K_c(&\boldsymbol{\varrho}_1, \boldsymbol{\varrho}_2, t_2)v(\boldsymbol{\varrho}_2)K_c(\boldsymbol{\varrho}_2, 0, t_3) + ..., \end{aligned} \tag{2.31}$$

where the matrix element K_c is defined by (2.27). The term $\frac{dW_1}{d\omega}$ in (2.30) corresponds to the first term (linear in v) in (2.31). Substituting (2.27) we have

$$\begin{aligned} \frac{dW_1}{d\omega} = &\frac{2\alpha}{\gamma^2}\mathrm{Re}\int\limits_0^\infty dt_1\int\limits_0^\infty dt_2 e^{-i(t_1+t_2)}\int d^2\varrho v(\boldsymbol{\varrho})\frac{q^2}{\pi^2\nu^2}\frac{1}{\sinh\nu t_1}\frac{1}{\sinh\nu t_2} \\ \times\exp&\left[-\frac{q\varrho^2}{\nu}(\coth\nu t_1 + \coth\nu t_2)\right]\left[R_1 + \frac{4q^2\varrho^2}{\nu^2\sinh\nu t_1\sinh\nu t_2}R_2\right], \end{aligned} \tag{2.32}$$

where $\nu = 2\sqrt{iq}$. Substituting in (2.32) the explicit expression for $v(\boldsymbol{\varrho})$ and integrating over $d^2\varrho$ and $d(t_1 - t_2)$ we obtain the following formula for the first correction to the probability of radiation

$$\begin{aligned}
\frac{dW_1}{d\omega} &= -\frac{\alpha}{4\pi\gamma^2 L}\text{Im}\ F(\nu); \quad F(\nu) = \int\limits_0^\infty \frac{dze^{-it}}{\sinh^2 z}\left[R_1 f_1(z) - 2iR_2 f_2(z)\right], \\
f_1(z) &= \left(\ln \varrho_c^2 + \ln \frac{\nu}{i} - \ln\sinh z - C\right) g(z) - 2\cosh z G(z), \\
f_2(z) &= \frac{\nu}{\sinh z}\left(f_1(z) - \frac{g(z)}{2}\right), \quad g(z) = z\cosh z - \sinh z, \\
G(z) &= \int\limits_0^z (1 - y\coth y)\, dy, \quad t = t_1 + t_2,\ z = \nu t
\end{aligned} \tag{2.33}$$

As it was said above (see (2.15), (2.19)), $\varrho_c = 1$ at

$$|\nu^2| = \nu_1^2 = 4QL_1 \leq 1 \quad (q = QL_1). \tag{2.34}$$

If the parameter $|\nu| > 1$, the value of ϱ_c is defined from the equation (2.15), where $\vartheta_1 \to \vartheta_2$, up to $\varrho_c = R_n/\lambda_c$. Then one has

$$\ln \varrho_c^2 + \ln \frac{\nu}{i} = \frac{1}{2}\ln(\varrho_c^4 4QL) - i\frac{\pi}{4} = -i\frac{\pi}{4}, \quad \varrho_c^4 4QL = 1. \tag{2.35}$$

It follows from (2.35) that expression (2.15) for ϱ_c^2, which we chose a priori, corresponds to the mean value of ϱ^2. From the above analysis we have that the factor at $g(z)$ in expression for $f_1(z)$ in (2.33) can be written in the form

$$(\ln \varrho_c^2 + \ln \frac{\nu}{i} - \ln\sinh z - C) \to (\ln \nu_0 \vartheta(1 - \nu_0) - i\frac{\pi}{4} - \ln\sinh z - C), \tag{2.36}$$

where

$$\nu_0^2 \equiv |\nu|^2 = 4q = 4QL(\varrho_c) = \frac{8\pi n Z^2\alpha^2\varepsilon\varepsilon'}{m^4\omega}L(\varrho_c), \tag{2.37}$$

$\vartheta(x)$ is the Heaviside step function. So, we have two representation of $|\nu|$ depending on ϱ_c: at $\varrho_c = 1$ it is $|\nu| = \nu_1$ while for $\varrho_c < 1$ it is $|\nu| = \nu_0$.

When a scattering is weak ($\nu_1 \ll 1$), the main contribution in (2.33) gives a region where $z \ll 1$. Then

$$\begin{aligned}
f_1(z) &\simeq -(C + \ln(it))\frac{z^3}{3} + \frac{2}{9}z^3 = \frac{z^3}{3}(\frac{2}{3} - C - \ln(it)), \\
-\text{Im}\ F(\nu) &= \frac{1}{9}\text{Im}\ \nu^2 \left(R_2 - R_1\right), \quad L \to L_1.
\end{aligned} \tag{2.38}$$

The corresponding asymptotes of the function $\Phi(\nu)$ (2.28) is

$$\Phi(\nu) \simeq \frac{\nu^2}{6}\left(R_1 + 2R_2\right), \quad (|\nu| \ll 1) \tag{2.39}$$

Combining the results obtained (2.38) and (2.39) we obtain the spectral distribution of the probability of radiation in the case when scattering is weak ($|\nu| \ll 1$)

$$\begin{aligned}
\frac{dW}{d\omega} &= \frac{dW_c}{d\omega} + \frac{dW_1}{d\omega} = \frac{\alpha}{2\pi\gamma^2}\text{Im}\left[\Phi(\nu) - \frac{1}{2L}F(\nu)\right] \\
&= \frac{\alpha}{2\pi\gamma^2}\frac{2Q}{3}\left[R_1\left(L_1 - \frac{1}{3}\right) + 2R_2\left(L_1 + \frac{1}{6}\right)\right] \\
&= \frac{4Z^2\alpha^3 n}{3m^2\omega}\left[\frac{\omega^2}{\varepsilon^2}\left(\ln\left(183Z^{-1/3}\right) - \frac{1}{6} - f(Z\alpha)\right)\right. \\
&\left. +2\left(1 + \frac{\varepsilon'^2}{\varepsilon^2}\right)\left(\ln\left(183Z^{-1/3}\right) + \frac{1}{12} - f(Z\alpha)\right)\right],
\end{aligned} \tag{2.40}$$

where L_1 is defined in (2.19). This expression coincide with the known Bethe-Heitler formula for probability of bremsstrahlung from high-energy electrons in the case of complete screening (if one neglects the contribution of atomic electrons) written down within power accuracy (omitted terms are of the order of powers of $\frac{1}{\gamma}$) with the Coulomb corrections, see e.g. Eq.(18.30) in [7], or Eq.(3.83) in [17].

The integral in the function Im $F(\nu)$ (2.33) which defines the first correction to the probability of radiation (2.33) can be transformed into the another form containing the real functions only

$$
\begin{aligned}
&-\mathrm{Im}\ F(\nu) = D_1(\nu_0)R_1 + \frac{1}{s}D_2(\nu_0)R_2; \quad s = \frac{1}{\sqrt{2}\nu_0}, \\
&D_1(\nu_0) = \int\limits_0^\infty \frac{dz e^{-sz}}{\sinh^2 z}\left[d(z)\sin sz + \frac{\pi}{4}g(z)\cos sz\right], \quad D_2(\nu_0) = \int\limits_0^\infty \frac{dz e^{-sz}}{\sinh^3 z} \\
&\times\left\{\left[d(z) - \frac{1}{2}g(z)\right](\sin sz + \cos sz) + \frac{\pi}{4}g(z)(\cos sz - \sin sz)\right\},
\end{aligned}
\tag{2.41}
$$

$$
d(z) = (\ln \nu_0 \vartheta(1-\nu_0) - \ln\sinh z - C)g(z) - 2\cosh z G(z),
$$

where the functions $g(z)$ and $G(z)$ are defined in (2.33). The form (2.41) is convenient for numerical calculations. Note, that parameter s in (2.41) is two times larger than used by Migdal [2].

At $\nu_0 \gg 1$ the function $F(\nu)$ (see (2.33) and (2.36)) can be written in the form

$$
F(\nu) = \int\limits_0^\infty \frac{dz}{\sinh^2 z}\left[R_1 f_1(z) - 2iR_2 f_2(z)\right]. \tag{2.42}
$$

Integrating over z we obtain

$$
-\mathrm{Im}\ F(\nu) = \frac{\pi}{4}R_1 + \frac{\nu_0}{\sqrt{2}}\left(\ln 2 - C + \frac{\pi}{4}\right)R_2. \tag{2.43}
$$

Under the same conditions ($\nu_0 \gg 1$) the function Im $\Phi(\nu)$ (2.28) is

$$
\mathrm{Im}\ \Phi(\nu) = \frac{\pi}{4}R_1 + \frac{\nu_0}{\sqrt{2}}R_2. \tag{2.44}
$$

Thus, at $\nu_0 \gg 1$ the relative contribution of the first correction $\frac{dW_1}{d\omega}$ is defined by

$$
r = \frac{dW_1}{dW_c} = \frac{1}{2L(\varrho_c)}\left(\ln 2 - C + \frac{\pi}{4}\right) \simeq \frac{0.451}{L(\varrho_c)}, \tag{2.45}
$$

where $L(\varrho_c) = \ln\frac{a_{s2}^2}{\lambda_c^2\varrho_c^2}$.

In the above analysis we did not consider an inelastic scattering of a projectile on atomic electrons. The potential $V_e(\boldsymbol{\varrho})$ connected with this process can be found from formula (2.10) by substitution $Z^2 \to Z, \vartheta_1 \to \vartheta_e = 0.153\vartheta_1$ (an analysis of an inelastic scattering on atomic electrons as well as the parameter ϑ_e can be found in [17]). The summary potential including both an elastic and an inelastic scattering is

$$
\begin{aligned}
&V(\boldsymbol{\varrho}) + V_e(\boldsymbol{\varrho}) = -Q(1+\frac{1}{Z})\boldsymbol{\varrho}^2\Big[\ln\gamma^2\vartheta_2^2 + \ln\frac{\boldsymbol{\varrho}^2}{4} + 2C \\
&+\frac{1}{Z+1}\left(\ln\frac{\vartheta_e^2}{\vartheta_1^2} - 2f\right)\Big] \\
&= -Q_{ef}\boldsymbol{\varrho}^2\Big(\ln\gamma^2\vartheta_{ef}^2 + \ln\frac{\boldsymbol{\varrho}^2}{4} + 2C\Big),
\end{aligned}
\tag{2.46}
$$

where

$$
Q_{ef} = Q\left(1+\frac{1}{Z}\right), \quad \vartheta_{ef} = \vartheta_1\exp\left[\frac{1}{1+Z}(Zf(\alpha Z) - 1.88) - \frac{1}{2}\right].
$$

3 An influence of the polarization of a medium

When one considers bremsstrahlung of enough soft photons $\omega \le \omega_0\gamma$, one has to take into account the effect of a polarization of the medium. In a dense medium the velocity of a photon propagation differs from the light velocity in the vacuum since the index of refraction $n(\omega) \neq 1$

$$n(\omega) = 1 - \frac{\omega_0^2}{2\omega^2}, \quad \omega_0^2 = \frac{4\pi n\alpha}{m}; \quad 1 - \frac{k}{\omega} \simeq \frac{1}{2}\left(1 - \frac{k^2}{\omega^2}\right) = \frac{1}{2}\frac{\omega_0^2}{\omega^2}. \tag{3.1}$$

Because of this the formation length diminishes as well as the probability of radiation (see [4], the qualitative discussion may be found in [6]). For analysis we use the general expression for the probability of radiation, see Eq.(2.1), [8]. The factor in front of exponent in this expression (see Eq.(2.2), [8]) contains two terms A and $\mathbf{B}$, the term A is not changed and the term $\mathbf{B}$ contains combination

$$\mathbf{v} - \frac{\mathbf{k}}{\omega} \simeq \boldsymbol{\vartheta} + \mathbf{n}\frac{\kappa_0^2}{2\gamma^2}, \quad \kappa_0 = \frac{\omega_0\gamma}{\omega}, \tag{3.2}$$

and its dependence on κ_0 (term of the order $1/\gamma^2$) may be neglected also. With regard for the polarization of a medium the formation length (2.1) acquires a form

$$l_f = \frac{2\gamma^2}{\omega}\left[1 + \gamma^2\vartheta^2 + \left(\frac{\gamma\omega_0}{\omega}\right)^2\right]^{-1}. \tag{3.3}$$

So, the dependence on ω_0 manifests itself in the exponent of Eq.(2.1), [8] and respectively in the exponent of (2.2) only:

$$a \to 2\frac{\omega\varepsilon}{\varepsilon - \omega}\left(1 - \frac{k}{\omega}v\right) \simeq a\kappa \equiv \tilde{a}, \quad \kappa \equiv 1 + \kappa_0^2. \tag{3.4}$$

Performing the substitution $a \to \tilde{a}$ in formula (2.7) we obtain for the potential (2.17)

$$\begin{aligned} V(\boldsymbol{\varrho}) \to \tilde{V}(\tilde{\boldsymbol{\varrho}}) &= \tilde{Q}\tilde{\boldsymbol{\varrho}}^2\left(L\left(\frac{\tilde{\varrho}}{2\sqrt{\kappa}}\right) - 2C\right) = \tilde{V}_c(\tilde{\boldsymbol{\varrho}}) + \tilde{v}(\tilde{\boldsymbol{\varrho}}), \quad \tilde{\varrho} = |\tilde{\boldsymbol{\varrho}}| = \varrho\sqrt{\kappa}, \\ \tilde{V}_c(\tilde{\boldsymbol{\varrho}}) &= \tilde{q}\tilde{\boldsymbol{\varrho}}^2, \quad \tilde{q} = \tilde{Q}\tilde{L}(\tilde{\varrho}_c), \quad \tilde{Q} = \frac{Q}{\kappa^2}, \quad \tilde{L}(\tilde{\varrho}_c) = \ln\frac{\kappa}{\gamma^2\vartheta_2^2\tilde{\varrho}_c^{\,2}}, \\ \tilde{v}(\tilde{\boldsymbol{\varrho}}) &= -\frac{\tilde{q}\tilde{\boldsymbol{\varrho}}^2}{\tilde{L}}\left(2C + \ln\frac{\tilde{\boldsymbol{\varrho}}^2}{4\tilde{\varrho}_c^{\,2}}\right). \end{aligned} \tag{3.5}$$

The substitution (3.4) in the expression for the probability of radiation (2.8) gives

$$R_1 \to R_1, \quad R_2 \to R_2\kappa \equiv \tilde{R}_2. \tag{3.6}$$

The value of the parameter $\tilde{\varrho}_c$ in (3.5) is determined by equation (compare with Eq.(2.35))

$$4\tilde{\varrho}_c^{\,4}\tilde{Q}\tilde{L}(\tilde{\varrho}_c) = 1, \quad \text{for} \quad 4\tilde{Q}\tilde{L}(1) \geq 1. \tag{3.7}$$

In the opposite case $\tilde{\varrho}_c = 1$ and this is possible in two intervals of the photon energy ω:

1. for $\kappa_0 \ll 1$ when the multiple scattering and effects of the polarization of a medium are weak;
2. for $\kappa_0 \gg 1$ when effects of the polarization of a medium become stronger then effects of the multiple scattering ($\nu_0 < \kappa$).

In an intermediate region we substitute $\tilde{\varrho}_c^{\,2} \to \varrho_c^2\kappa$ in Eq.(3.7). After it we obtain the equation for ϱ_c which coincides with Eq.(2.35), see also (2.37):

$$\frac{1}{\varrho_c^4} = \nu_0^2(\varrho_c), \quad \nu_0^2(\varrho_c) = 4QL(\varrho_c). \tag{3.8}$$

Thus, for $\tilde{\varrho}_c < 1$ we have

$$\tilde{\nu}_0 = \sqrt{4\tilde{Q}\tilde{L}(\tilde{\varrho}_c)} = \frac{1}{\tilde{\varrho}_c^{\,2}} = \frac{1}{\varrho_c^2\kappa} = \frac{\nu_0}{\kappa}, \quad \tilde{L}(\tilde{\varrho}_c) = L(\varrho_c), \tag{3.9}$$

while for $\tilde{\nu}_0 < 1$ we have

$$\tilde{\nu}_0 = \sqrt{4\tilde{Q}\tilde{L}(1)} = \frac{2}{\kappa}\sqrt{Q}\ln\left(\frac{a_{s2}^2\kappa}{\lambda_c^2}\right). \tag{3.10}$$

The spectral distribution of the probability of radiation (2.40) with allowance for polarization of a medium have the form

$$\frac{dW}{d\omega} = \frac{\alpha}{2\pi\gamma^2}\text{Im}\left[\tilde{\Phi}(\tilde{\nu}) - \frac{1}{2\tilde{L}(\tilde{\varrho}_c)}\tilde{F}(\tilde{\nu})\right], \tag{3.11}$$

where

$$\tilde{\Phi}(R_1, R_2) = \Phi(R_1, \tilde{R}_2), \quad \tilde{F}(R_1, R_2) = F(R_1, \tilde{R}_2),$$

We consider now the case when an influence the polarization of a medium manifests itself in the conditions of the strong LPM effect ($\nu_0 \gg 1$). This influence becomes essential for low energy photons, when the mean square angle of the multiple scattering (2.15) on the formation length of a photon becomes smaller than ω_0^2/ω^2 $\left(\tilde{\nu}_0 = \dfrac{\nu_0}{\kappa} \leq 1, \quad \kappa_0^2 \gg 1\right)$. Indeed, in the case $\tilde{\nu}_0 \gg 1 (\nu_0 \gg \kappa_0^2)$ one can use asymptotes of functions $\Phi(\nu)$ and $F(\nu)$ at $\nu_0 \gg 1$ (see (2.42), (2.44)), we have

$$\begin{gathered}\frac{dW}{d\omega} = \frac{\alpha}{2\pi\gamma^2}R_2\kappa\frac{\tilde{\nu}_0(1+\tilde{r})}{\sqrt{2}} = \frac{\alpha}{2\pi\gamma^2}R_2\frac{\nu_0}{\sqrt{2}}(1+r), \\ \tilde{r} = \frac{0.451}{\tilde{L}(\tilde{\varrho}_c)}, \quad \tilde{L}(\tilde{\varrho}_c) = L(\varrho_c) = \ln\frac{a_{s2}^2}{\lambda_c^2\varrho_c^2}.\end{gathered} \tag{3.12}$$

In the opposite case $\nu_0 \ll \kappa_0^2$, the characteristic momentum transfer in the used units (ζ_c) are defined by value $\kappa_0^2 (\tilde{\varrho}_c^2 = 1)$, one can use asymptotic expansions (2.38) and (2.39) and we have for the spectral distribution of the probability of radiation

$$\frac{dW}{d\omega} = \frac{16}{3}\frac{Z^2\alpha^3 n}{m^2\omega\kappa_0^2}\left(L_p + \frac{1}{12} - f(Z\alpha)\right) = \frac{4}{3\pi}\frac{Z^2\alpha^2\omega}{m\gamma^2}\left(L_p + \frac{1}{12} - f(Z\alpha)\right), \tag{3.13}$$

where $f(Z\alpha)$ is defined in (A.10), $L_p = \ln\left(183Z^{-1/3}\kappa_0\right)$. The results obtained agree with given in [4] where calculations are fulfilled within logarithmic accuracy and without Coulomb corrections. It is seen that a dependence of spectral distribution on photon energy ($\omega d\omega$) differs essentially from the Bethe-Heitler one ($d\omega/\omega$), the probability is independent on density n.

The formula (3.13) is applicable only up to value $\kappa_0 = \lambda_c/R_n$ or if $\omega > \omega_b$, where

$$\omega_b = \frac{R_n}{\lambda_c}\omega_0\gamma \simeq \alpha Z^{1/3}\gamma\omega_0; \quad \frac{\omega}{\varepsilon} > \alpha Z^{1/3}\frac{\omega_0}{m}. \tag{3.14}$$

For example, for electrons with energy $\varepsilon = 25\ GeV$ and gold target ($\omega_0 = 80\ eV$) one has $\omega_b \simeq 125\ KeV$. For $\omega < \omega_b$ one has take into account the form factor of a nucleus (see Appendix B). In this case the argument of the logarithm in (3.14) ceases its dependence on photon energy ω. In the limit $\omega \ll \omega_b$ the spectral distribution of the probability of radiation is

$$\frac{dW}{d\omega} = \frac{4}{3\pi}\frac{Z^2\alpha^2\omega}{m\gamma^2}\left(\ln\frac{a_s}{R_n} - 0.02\right) \tag{3.15}$$

4 A target of a finite thickness

In the case when a finiteness of a target is essential the probability of radiation is defined not only by the relative time $\tau = t_2 - t_1$ as in Section 2. The used radiation theory is formulated in terms of two times (see eqs.(2.1) - (2.3) of [8]). Proceeding from this formulation we can obtain more general expression which takes into account boundary effects. With allowance for polarization of a medium we have for the spectral distribution of the probability of radiation

$$\begin{gathered}\frac{dw}{d\omega} = \frac{4\alpha}{\omega}\text{Re}\int\limits_{-\infty}^{\infty}dt_2\int\limits_{-\infty}^{t_2}dt_1\exp\left(-i\mu(t_2)t_2 + i\mu(t_1)t_1\right) \\ \times\left[r_1\varphi_0(0,t_2,t_1) - ir_2\boldsymbol{\nabla}\boldsymbol{\varphi}(0,t_2,t_1)\right],\end{gathered} \tag{4.1}$$

where

$$\mu(t) = \vartheta(-t) + \vartheta(T-t) + \kappa\vartheta(t)\vartheta(T-t), \quad T = \frac{la}{2} = \frac{l\omega m^2}{2\varepsilon\varepsilon'},$$
$$r_1 = \frac{\omega^2}{\varepsilon^2}, \quad r_2 = 1 + \frac{\varepsilon'^2}{\varepsilon^2}, \quad \kappa = 1 + \kappa_0^2, \tag{4.2}$$

here l is the thickness of a target,

κ_0 is defined in (3.2). So, we split time interval (in the used units) into three parts: before target ($t < 0$), after target ($t > T$) and inside target ($0 \leq t \leq T$). The functions $\varphi_\mu(\boldsymbol{\varrho}, t_2, t_1), \quad \varphi_\mu = \varphi_\mu(\varphi_0, \boldsymbol{\varphi})$ satisfy the equation (2.9), but now the potential V depends on time

$$i\frac{\partial\varphi_\mu}{\partial t} = \mathcal{H}(t)\varphi_\mu, \quad \mathcal{H}(t) = \mathbf{p}^2 - iV(\boldsymbol{\varrho})g(t), \quad g(t) = \vartheta(t)\vartheta(T-t);$$
$$\varphi_0(\boldsymbol{\varrho}, t_1, t_1) = \delta(\boldsymbol{\varrho}), \quad \boldsymbol{\varphi}(\boldsymbol{\varrho}, t_1, t_1) = \mathbf{p}\delta(\boldsymbol{\varrho}). \tag{4.3}$$

Using an operator form of a solution of Eq. (4.3) (compare with (2.11)) we can present the probability (4.1) in the form

$$\frac{dw}{d\omega} = \frac{4\alpha}{\omega}\text{Re}\int\limits_{-\infty}^{\infty} dt_2 \int\limits_{-\infty}^{t_2} dt_1 \exp\left(-i\mu(t_2)t_2 + i\mu(t_1)t_1\right)$$
$$\times \left\langle 0|r_1 S(t_2, t_1) + r_2\mathbf{p}S(t_2, t_1)\mathbf{p}|0\right\rangle, \quad S(t_2, t_1) = \text{T}\exp\left[-i\int\limits_{t_1}^{t_2}\mathcal{H}(t)dt\right], \tag{4.4}$$

where the symbol T means the chronological product. Note, that in (4.1) and (4.4) it is implied that subtraction is made at $V = 0$, $\mu(t) = 1$ ($\kappa = 1$).

Integrals over time in (4.4) we present as integrals over four domains:

1. $t_1 \leq 0, \ 0 \leq t_2 \leq T$;
2. $0 \leq t_1 \leq T, \ 0 \leq t_2 \leq T$;
3. $0 \leq t_1 \leq T, \ t_2 \geq T$;
4. $t_1 \leq 0, t_2 \geq T$;

in two more domains $t_{1,2} \leq 0$ and $t_{1,2} \geq T$ an electron is moving entirely free and there is no radiation. We consider in this Section the case, when the thickness of a target L is much larger than formation length l_f (3.3) or $(\nu_0 + \kappa)T \gg 1$. In this case domain 4) doesn't contribute. The contributions of other domains are

$$I_1 \simeq \int\limits_{-\infty}^{0} dt_1 \int\limits_{0}^{\infty} dt_2 \exp\left(i(t_1 - \kappa t_2)\right)\exp\left(-iHt_2\right)\exp\left(iH_0t_1\right) = -\frac{1}{H+\kappa}\frac{1}{H_0+1},$$
$$I_2 = \int\limits_{0}^{T} dt_2 \int\limits_{0}^{t_2} dt_1 \exp\left(-i(H+\kappa)(t_2 - t_1)\right) \simeq T\int\limits_{0}^{\infty} d\tau \exp\left(-i(H+\kappa)\tau\right)$$
$$-\int\limits_{0}^{\infty} \tau d\tau \exp\left(-i(H+\kappa)\tau\right) = -i\frac{T}{H+\kappa} + \frac{1}{(H+\kappa)^2}, \quad I_3 \simeq -\frac{1}{H_0+1}\frac{1}{H+\kappa}, \tag{4.5}$$

where $H_0 = \mathbf{p}^2$. The term in I_2: $-iT/(H+\kappa)$ describes the probability of radiation considered in previous Sections. All other terms define the probability of radiation of boundary photons [1]. So, making

[1]Radiation of boundary photons in an inhomogeneous electromagnetic field was considered in [18].

mentioned subtraction we have for the spectral distribution of the probability of radiation of boundary photons

$$\begin{aligned}
&\frac{dw_b}{d\omega}=\frac{4\alpha}{\omega}\text{Re}\left\langle 0|r_1M+r_2\mathbf{p}M\mathbf{p}|0\right\rangle, \quad M=M_V^{(1)}+M_V^{(2)}+M_0; \\
&M_V^{(1)}=\frac{2}{(\mathbf{p}^2+\kappa)(\mathbf{p}^2+1)}-\frac{2}{(H+\kappa)(\mathbf{p}^2+1)}, \\
&M_V^{(2)}=\frac{1}{(H+\kappa)^2}-\frac{1}{(\mathbf{p}^2+\kappa)^2}, \\
&M_0=\frac{1}{(\mathbf{p}^2+1)^2}-\frac{2}{(\mathbf{p}^2+\kappa)(\mathbf{p}^2+1)}+\frac{1}{(\mathbf{p}^2+\kappa)^2},
\end{aligned} \tag{4.6}$$

For a convenience here we made the subtraction in two stages: first (in $M_V^{(1)}$, $M_V^{(2)}$) we subtracted terms with $V=0$ and second (in M_0) we subtracted terms with both $V=0$ and $\kappa=1$.

We consider important case when both the LPM effect and the polarization of a medium are essential. We will calculate the main term with $V(\boldsymbol{\varrho})=V_c(\boldsymbol{\varrho})$, see (2.17). Needed combinations are

$$\begin{aligned}
&\left\langle 0\left|\frac{1}{H+\kappa}\frac{1}{\mathbf{p}^2+1}\right|0\right\rangle \\
&=-\int\limits_0^\infty dt_1\int\limits_0^\infty dt_2\exp\left(-i(t_1+\kappa t_2)\right)\int d^2\varrho K_c(0,\boldsymbol{\varrho},t_2)K_0(\boldsymbol{\varrho},0,t_1), \\
&\left\langle 0\left|\frac{1}{(H+\kappa)^2}\right|0\right\rangle=-\int\limits_0^\infty tdt\exp(-i\kappa t)K_c(0,0,t), \quad \langle 0|M_0|0\rangle=\frac{1}{4\pi}, \\
&\langle 0|\mathbf{p}M_0\mathbf{p}|0\rangle=\frac{\pi}{(2\pi)^2}\int\limits_0^\infty dp^2p^2M_0=\frac{1}{4\pi}\left[\left(1+\frac{2}{\kappa-1}\right)\ln\kappa-2\right],
\end{aligned} \tag{4.7}$$

where the functions $K_0(\boldsymbol{\varrho}_2,\boldsymbol{\varrho}_1,t)$ and $K_c(\boldsymbol{\varrho}_2,\boldsymbol{\varrho}_1,t)$ are defined in (2.24) and (2.27). Substituting into (4.7) the explicit expressions for these functions, calculating the vector derivatives as indicated in (4.6) we have for contribution of the first term in (4.7)

$$\begin{aligned}
&\frac{dw_b^{(1)}}{d\omega}=-\frac{2\alpha}{\pi\omega}r_2\ \text{Re}\ \nu^2\int\limits_0^\infty dt_1\int\limits_0^\infty dt_2\exp\left(-i(t_1+\kappa t_2)\right) \\
&\times\left[\frac{1}{\left(\sinh\nu t_2+\nu t_1\cosh\nu t_2\right)^2}-\frac{1}{\left(\nu t_1+\nu t_2\right)^2}\right] \\
&=-\frac{2\alpha}{\pi\omega}r_2\ \text{Im}\ \nu\int\limits_0^\infty dt_1\int\limits_0^\infty dt_2\exp\left(-i(t_1+t_2)\right)\left[\frac{1}{\tanh\tilde{\nu}t_2+\nu t_1}-\frac{1}{\tilde{\nu}t_2+\nu t_1}\right],
\end{aligned} \tag{4.8}$$

where $\tilde{\nu}=\nu/\kappa$, the second term in the square brackets is the subtraction term in accordance with (4.6) (the term $M_V^{(1)}$). For practical use it is convenient to write the probability (4.8) using real variables. After some transformations it can be written as

$$\begin{aligned}
&\frac{dw_b^{(1)}}{d\omega}=\frac{2\alpha}{\pi\omega}r_2\int\limits_0^\infty dt\exp(-t)\left(\cos t+\sin t\right)\int\limits_0^t dy\left[\frac{1}{t-y+s\tanh\left(y/\kappa s\right)}\right. \\
&\left.-\frac{1}{t-y+y/\kappa}\right],
\end{aligned} \tag{4.9}$$

where $s=\dfrac{1}{\sqrt{2}\nu_0}$, parameter ν_0 is defined in (2.37). Repeating the same operations with the second term in (4.7) (this is the contribution of the term $M_V^{(2)}$ in (4.6)) we have

$$\begin{aligned}\frac{dw_b^{(2)}}{d\omega} &= \frac{\alpha}{\pi\omega}r_2\ \mathrm{Re}\ \nu^2\int\limits_0^\infty tdt\exp(-i\kappa t)\left[\frac{1}{\sinh^2\nu t}-\frac{1}{(\nu t)^2}\right] \\ &= \frac{\alpha}{\pi\omega}r_2\ \mathrm{Re}\int\limits_0^\infty dz\exp\left(-i\frac{z}{\tilde\nu}\right)\left[\frac{z}{\sinh^2 z}-\frac{1}{z}\right] \\ &= \frac{\alpha}{\pi\omega}r_2\int\limits_0^\infty dz\exp(-\tilde s z)\cos\tilde s z\left[\frac{z}{\sinh^2 z}-\frac{1}{z}\right],\end{aligned} \tag{4.10}$$

where $z=\nu t,\ \tilde s=\dfrac{1}{\sqrt2\tilde\nu_0},\ \tilde\nu_0=\nu_0/\kappa$. The contribution of the term M_0 in (4.6) is calculated in (4.7)

$$\frac{dw_b^{(3)}}{d\omega}=\frac{\alpha}{\pi\omega}\left\{r_1+r_2\left[\left(1+\frac{2}{\kappa-1}\right)\ln\kappa-2\right]\right\}. \tag{4.11}$$

The complete expression for the spectral distribution of the probability of radiation of boundary photons, in the case when both the LPM effect and the polarization of a medium are taken into account, is

$$\frac{dw_b}{d\omega}=\sum_{k=1}^{3}\frac{dw_b^{(k)}}{d\omega}. \tag{4.12}$$

We consider now the limiting case when LPM effect is very strong ($\tilde\nu_0\gg1$). In this case we find for probabilities in formulae (4.9) and (4.10)

$$\begin{aligned}\frac{dw_b^{(1)}}{d\omega} &= \frac{2\alpha}{\pi\omega}r_2\left[\ln\tilde\nu_0-C-\frac{\ln\kappa}{\kappa-1}+\frac{\pi^2}{8\sqrt2\tilde\nu_0}+\frac{1}{\sqrt2\tilde\nu_0}\left(\ln\tilde\nu_0+1-C+\frac{\pi}{4}\right)\right], \\ \frac{dw_b^{(2)}}{d\omega} &= \frac{\alpha}{\pi\omega}r_2\left[1-\ln2\tilde\nu_0+C-\frac{\pi^2}{6\sqrt2\tilde\nu_0}\right].\end{aligned} \tag{4.13}$$

Substituting asymptotes obtained and (4.11) into (4.12) we have

$$\frac{dw_b}{d\omega}=\frac{\alpha}{\pi\omega}\left\{r_1+r_2\left[\ln\nu_0-1-C-\ln2+\frac{\sqrt2}{\nu_0}\left(\kappa\frac{\pi^2}{24}+\ln\nu_0+1-C\frac{\pi}{4}\right)\right]\right\}. \tag{4.14}$$

As one can expect, the probability of radiation at $\nu_0\gg1+\kappa_0^2$ depends on the polarization of a medium in the term $\propto1/\nu_0$ only.

In the opposite case $\tilde\nu_0\ll1(\nu_0\gg1)$, the probabilities $dw_b^{(1)},\ dw_b^{(2)}\propto\tilde\nu_0^4$ and probability of radiation of boundary photons is determined by the polarization of a medium. Just in this case radiation of boundary photons is known as the transition radiation:

$$\frac{dw_b}{d\omega}\simeq\frac{dw_b^{(3)}}{d\omega}=\frac{\alpha}{\pi\omega}\left\{r_1+r_2\left[\left(1+\frac{2}{\kappa-1}\right)\ln\kappa-2\right]\right\}. \tag{4.15}$$

In the case of weak LPM effect $\nu_1\ll1$ (see (2.34), $\omega\ll\varepsilon$) we have

$$\frac{dw_b}{d\omega}\simeq\frac{\alpha}{\pi\omega}r_2\left(-\frac{2}{21}\nu_1^4\right). \tag{4.16}$$

In this case what we calculated as the boundary photons contribution is actually correction (very small) to the probability $l\dfrac{dW}{d\omega}$ (2.40) which in this case has additional (suppression) factor $1-\dfrac{16}{21}\nu_1^4$ which follows from the decomposition of the function Im Φ.

The LPM effect for the case of structured targets (with many boundaries) was analyzed recently in [19]. The radiation of the boundary photons with regard for the multiple scattering was considered in [20] (for $\omega\ll\varepsilon$), the polarization of a medium was added in [21] and [22]. Our results, which are consistent with obtained [21], are presented in more convenient for application form and the Coulomb corrections are included. In these papers the probability of radiation of boundary photons (under condition of applicability of Eq.(4.14)) was analyzed also to within the logarithmic accuracy (see Eq.(20) in [21] and Eq.(15 in [22])). This accuracy is insufficient for parameters connected with experiment [12]-[14]. For example, for $\varepsilon=25\ GeV$ and heavy elements the value ν_0 equates κ for $\nu_0\sim20$. One can see from Eq.(4.14) that in this case $\ln\nu_0$ is nearly completely compensated by constant terms.

5 A thin target

Finally we consider a situation when the formation length of radiation is much larger than the thickness l of a target (a thin target, $l_c \gg l$). In this case the radiated photon is propagating in the vacuum and one can neglect the polarization of a medium.

Operator $S(t_1, t_2)$ (4.4) we present in the form

$$
\begin{aligned}
S(t_2, t_1) &= T\exp\left[-i\int_{t_1}^{t_2} \mathcal{H}(t)dt\right] = \exp\left(-iH_0t_2\right)\mathcal{L}(t_2, t_1)\exp\left(iH_0t_1\right);\\
\mathcal{L}(t_2, t_1) &= \exp\left(iH_0t_2\right)S(t_2, t_1)\exp\left(-iH_0t_1\right).
\end{aligned}
\tag{5.1}
$$

Differentiating the operator $\mathcal{L}(t_2, t_1)$ over the first of arguments we obtain

$$
\frac{\partial\mathcal{L}(t, t_1)}{\partial t} = -\exp\left(iH_0t\right)V(\boldsymbol{\varrho}, t)S(t, t_1)\exp\left(-iH_0t_1\right) = -V(\boldsymbol{\varrho} + 2\mathbf{p}t, t)\mathcal{L}(t, t_1),
\tag{5.2}
$$

where $V(\boldsymbol{\varrho}, t) = V(\boldsymbol{\varrho})g(t)$ (see (2.9), (4.3)). The formal solution of this equation with the initial condition $\mathcal{L}(t_1, t_1) = 1$ has the form

$$
\mathcal{L}(t_2, t_1) = \mathrm{T}\exp\left[-\int_{t_1}^{t_2} dtV(\boldsymbol{\varrho} + 2\mathbf{p}t, t)\right],
\tag{5.3}
$$

where T means the chronological product. This solution is exact. Now we take into account that we are considering a short characteristic time contributing into integral (5.3), or more precisely

$$
t \leq T = \frac{la}{2}, \quad l \ll l_c = \frac{2}{a\zeta}, \quad T \ll \frac{1}{\zeta},
\tag{5.4}
$$

where l_c, ζ are defined in (2.1). Since the main contribution give $p \sim \sqrt{\zeta}$,
$\varrho \sim 1/\sqrt{\zeta}$, $pt \ll 1/\sqrt{\zeta} \sim \varrho$,where p is characteristic mean value of operator $|\mathbf{p}|$, one can neglect by the term $2\mathbf{p}t$ in (5.3), so that

$$
\mathcal{L}(t_2, t_1) \simeq \exp\left[-\int_{t_1}^{t_2} dtV(\boldsymbol{\varrho}, t)\right].
\tag{5.5}
$$

In the probability of radiation enters the expression (cp (2.12), (4.4))

$$
\begin{aligned}
&\left\langle 0\left|\exp\left(-iH_0t_2\right)(\mathcal{L} - 1)\exp\left(iH_0t_1\right)\right|0\right\rangle\\
&= \int d^2\varrho\,(\mathcal{L} - 1)\left\langle 0\left|\exp\left(-iH_0t_2\right)\right|\boldsymbol{\varrho}\right\rangle\left\langle\boldsymbol{\varrho}\left|\exp\left(iH_0t_1\right)\right|0\right\rangle.
\end{aligned}
\tag{5.6}
$$

Using an explicit form (2.24) of the matrix element $\left\langle 0\left|\exp\left(-iH_0t_2\right)\right|\boldsymbol{\varrho}\right\rangle$ and neglecting terms of the order $\sim T(l/l_c)$ one obtains starting from (4.4) for the spectral distribution of the probability of radiation

$$
\begin{aligned}
\frac{dw_{th}}{d\omega} &= \frac{\alpha}{4\pi^2\omega}\int_{-\infty}^{0}\frac{dt_1}{t_1}\int_{0}^{\infty}\frac{dt_2}{t_2}\int d^2\varrho\,(r_1 + r_2\mathbf{p}_1\mathbf{p}_2)\\
&\times\exp\left[-i(t_2 - t_1) + i\frac{\boldsymbol{\varrho}^2}{4}\left(\frac{1}{t_2} - \frac{1}{t_1}\right)\right](\exp(-VT) - 1)\\
&= \frac{\alpha}{\pi^2\omega}\int d^2\varrho\left[r_1K_0^2(\varrho) + r_2K_1^2(\varrho)\right](1 - \exp(-VT)),
\end{aligned}
\tag{5.7}
$$

where $\mathbf{p}_1$ ($\mathbf{p}_2$) is the operator $\mathbf{p} = -i\boldsymbol{\nabla}$ acting on the function of $\boldsymbol{\varrho}^2/t_1$ ($\boldsymbol{\varrho}^2/t_2$), K_n is the modified Bessel function. Here we took into account that in our case contribute domain $|t_1|, |t_2| \gg T$ and $t_1 \leq 0, t_2 \geq 0$ since in domains $t_{1,2} \leq 0$ and $t_{1,2} \geq T$ an electron is moving entirely free and there is no radiation. In implicit form the factorization contained in (5.7) is presented in [23]. If
$V(\varrho = 1)T \ll 1$ one can expand the exponent (the contribution of the region $\varrho \gg 1$ is exponentially

damped because in this region $K_{0,1}(\varrho) \propto \exp(-\varrho)$). In the first order over VT using the explicit expression for the potential (2.10) we have to calculate following integrals:

$$\begin{gathered}
\int\limits_0^\infty K_0^2(\varrho)\varrho^3 d\varrho = \frac{1}{3}, \quad \int\limits_0^\infty K_0^2(\varrho) \ln \varrho \varrho^3 d\varrho = \frac{1}{3}\left(\ln 2 - C + \frac{1}{6}\right); \\
\int\limits_0^\infty K_1^2(\varrho)\varrho^3 d\varrho = \frac{2}{3}, \quad \int\limits_0^\infty K_1^2(\varrho) \ln \varrho \varrho^3 d\varrho = \frac{2}{3}\left(\ln 2 - C - \frac{1}{12}\right)
\end{gathered} \tag{5.8}$$

Substituting these integrals one obtains in this case the Bethe-Heitler formula with the Coulomb corrections (2.40).

We analyze now the opposite case when the multiple scattering of a particle traversing a target is strong ($V(\varrho = 1)T \gg 1$, the mean square of multiple scattering angle $\vartheta_s^2 \gg 1/\gamma^2$). We present the function $V(\varrho)T$ (see (2.9), (2.10) and (2.19)) as

$$\begin{gathered}
V(\varrho)T = \frac{\pi Z^2\alpha^2 nl}{m^2}\varrho^2\left(\ln\frac{4a_{s2}^2}{\lambda_c^2\varrho^2} - 2C\right) = A\varrho^2 \ln\frac{\chi_t}{\varrho^2} = A\varrho^2\left(\ln\frac{\chi_t}{\varrho_t^2} - \ln\frac{\varrho^2}{\varrho_t^2}\right) \\
= k\varrho^2\left(1 - \frac{1}{L_t}\ln\frac{\varrho^2}{\varrho_t^2}\right); \quad A\varrho_t^2 \ln\frac{\chi_t}{\varrho_t^2} = 1, \quad L_t = \ln\frac{\chi_t}{\varrho_t^2} \simeq \ln\frac{4a_{s2}^2}{\lambda_c^2\varrho_t^2},
\end{gathered} \tag{5.9}$$

where ϱ_t is the lower boundary of values contributing into the integral over ϱ. Substituting this expression into (5.7) we have the integral

$$2\pi\int\limits_0^\infty \varrho d\varrho K_1^2(\varrho)\left\{1 - \exp\left[-k\varrho^2\left(1 - \frac{1}{L_t}\ln\frac{\varrho^2}{\varrho_t^2}\right)\right]\right\} \equiv \pi J. \tag{5.10}$$

In this integral we expand the exponent in the integrand over $1/L_t$ keeping the first term of the expansion. We find

$$\begin{aligned}
& J = J_1 + J_2, \quad J_1 = 2\int\limits_0^\infty K_1^2(\varrho)\left[1 - \exp\left(-k\varrho^2\right)\right]\varrho d\varrho \\
& = 2k\int\limits_0^\infty d\varrho\varrho^3\left[K_0(\varrho)K_2(\varrho) - K_1^2(\varrho)\right]\exp\left(-k\varrho^2\right), \\
& J_2 = -\frac{2k}{L_t}\int\limits_0^\infty K_1^2(\varrho)\exp\left(-k\varrho^2\right)\ln\frac{\varrho^2}{\varrho_t^2}\varrho^3 d\varrho
\end{aligned} \tag{5.11}$$

In the integral J_1 we performed an integration by parts. In the integrals in (5.11) it is convenient to substitute $z = k\varrho^2$ then

$$\begin{aligned}
& J_1 = \frac{1}{k}\int\limits_0^\infty \left[K_0\left(\sqrt{\frac{z}{k}}\right)K_2\left(\sqrt{\frac{z}{k}}\right) - K_1^2\left(\sqrt{\frac{z}{k}}\right)\right]\exp(-z)zdz, \\
& J_2 = -\frac{1}{kL_t}\int\limits_0^\infty K_1^2\left(\sqrt{\frac{z}{k}}\right)\exp(-z)\ln z z dz.
\end{aligned} \tag{5.12}$$

Expanding the modified Bessel functions $K_n(x)$ at $x \ll 1$ and taking the integrals in the last expression we have

$$\begin{aligned}
& J = J_1 + J_2 = \left(1 + \frac{1}{2k}\right)(\ln 4k - C) + \frac{1}{2k} - 1 + \frac{C}{L_t}, \\
& k = \frac{\pi Z^2\alpha^2}{m^2} nl\left(L_t + 1 - 2C\right).
\end{aligned} \tag{5.13}$$

In the term with K_0^2 in (5.7) the region $\varrho \sim 1$ contributes. So we have

$$J_3 = 2\int\limits_0^\infty K_0^2(\varrho)\left(1 - \exp(-VT)\right)\varrho d\varrho \simeq 2\int\limits_0^\infty K_0^2(\varrho)\varrho d\varrho = 1. \tag{5.14}$$

Substituting found J and J_3 into (5.7) we obtain for the spectral distribution of the probability of radiation in a thin target at conditions of the strong multiple scattering

$$\frac{dw_{th}}{d\omega} = \frac{\alpha}{\pi\omega}\left(r_1 + r_2 J\right). \tag{5.15}$$

The logarithmic term in this formula is well known in theory of the collinear photons radiation at scattering of a radiating particle on angle much larger than characteristic angles of radiation $\sim 1/\gamma$. It is described with logarithmic accuracy in a quasi-real electron approximation (see [24], Appendix B2).

The formula (5.7) presents the probability of radiation in the case when the formation length $l_c \gg l$. It is known,see e.g. [7], that in this case a process of scattering of a particle is independent of a radiation process and a differential probability of radiation at scattering with the momentum transfer $\mathbf{q}$ can be presented in the form

$$dW_\gamma = dw_s(\mathbf{q})dw_r(\mathbf{q},\mathbf{k}), \tag{5.16}$$

where $dw_s(\mathbf{q})$ is the differential probability of scattering with the momentum transfer $\mathbf{q}$ which depends on properties of a target. The function $dw_r(\mathbf{q},\mathbf{k})$ is the probability of radiation of a photon with a momentum $\mathbf{k}$ when an emitting electron acquires the momentum transfer $\mathbf{q}$. This probability has a universal form which is independent of properties of a target. For an electron traversing an amorphous medium this fact is reflected in formula (5.7). Indeed, passing on to a momentum space we have

$$\begin{aligned} dw_r(\mathbf{q},\mathbf{k}) &= \frac{\alpha d\omega}{\pi^2\omega}\int d^2\varrho \left[r_1 K_0^2(\varrho) + r_2 K_1^2(\varrho)\right]\left(1-\exp(-i\mathbf{q}\boldsymbol{\varrho})\right) \\ &= \frac{\alpha d\omega}{\pi\omega}\left[r_1 F_1\left(\frac{q}{2}\right) + r_2 F_2\left(\frac{q}{2}\right)\right]; \\ F_1(x) &= 1 - \frac{\ln\left(x+\sqrt{1+x^2}\right)}{x\sqrt{1+x^2}}, \quad F_2(x) = \frac{2x^2+1}{x\sqrt{1+x^2}}\ln\left(x+\sqrt{1+x^2}\right) - 1. \end{aligned} \tag{5.17}$$

Remind that q is measured in electron mass. The probability of radiation in this form was found in [21]. For a differential probability of scattering (here we consider the multiple scattering) there is a known formula (cp (2.5), (2.6) and (2.9))

$$\begin{aligned} dw_s(\mathbf{q}) &= F_s(\mathbf{q})d^2q, \quad F_s(\mathbf{q}) = \frac{1}{(2\pi)^2}\int d^2\varrho \exp\left(-i\mathbf{q}\boldsymbol{\varrho}\right)\exp\left(-V_s(\varrho)l\right), \\ V_s(\varrho) &= n\int d^2q\left(1-\exp(-i\mathbf{q}\boldsymbol{\varrho})\right)\sigma(\mathbf{q}), \end{aligned} \tag{5.18}$$

where $\sigma(\mathbf{q})$ is the cross section of single scattering.

Using the formula (5.17) one can easily obtain to within logarithmic accuracy expressions (5.15),(4.14). Both a radiation of boundary photons and a radiation in a thin target may be considered as a radiation of collinear photons (see e.g. [24]) in the case when an emitting particle deviates at large angle ($\vartheta_s \gg 1/\gamma, q \gg 1$). Using (5.17) at $x \gg 1$ we find

$$\begin{aligned} dw_r(q) &\simeq \frac{\alpha d\omega}{\pi\omega}\left[r_1 + r_2\left(\ln q^2 - 1\right)\right]; \\ \int d^2q dw_r(q)F_s(\mathbf{q}) &\simeq \frac{\alpha d\omega}{\pi\omega}\left[r_1 + r_2\left(\ln\overline{q^2} - 1\right)\right]. \end{aligned} \tag{5.19}$$

For a thin target value of $\overline{q^2}$ is defined by mean square of multiple scattering angle on a thickness of a target l, and for boundary photons is the same but on the formation length l_f. However, if we one intends to perform computation beyond a logarithmic accuracy, the method given in this Section has advantage since there is no necessity to calculate $F_s(\mathbf{q})$ and in our approach a problem of calculation of the Coulomb corrections is solved in a rather simple way.

6 A target of an intermediate thickness $l \sim l_c$

It appears that used in Section 4 approach permits one to consider an important case when $l \sim l_c$.

According to the partition of integrals over time in formula (4.4) into four domains we can write the probability of radiation as

$$\frac{dw}{d\omega} = \sum_{n=1}^{4} \frac{dw_n}{d\omega}, \quad \frac{dw_n}{d\omega} = \frac{4\alpha}{\omega}\mathrm{Re}\left(r_1 I_n^{(1)} + r_2 I_n^{(2)}\right). \tag{6.1}$$

The integrals in $I_n^{(1,2)}$ we compute on the assumption: $\nu_0 \gg 1,\ T \ll 1,\ \nu_0 T \sim 1,\ \kappa = 1$. Since integrals in $I_n^{(1)}$ don't contain the logarithmic divergence, only the domain 4 contributes. In the domains 1-3 one of the integrals in $I_n^{(1)}$ contains an integration over an interval $0 \le t \le T$ and due to this reason $dw_{1,2,3} \propto T \ll 1$. So, we consider $I_4^{(1)}$

$$\begin{aligned} I_4^{(1)} &= \int\limits_{-\infty}^{0} dt_1 \int\limits_{T}^{\infty} dt_2 \exp(-i(t_1+t_2))\langle 0 \big| \exp(-i(H_0(t_2-T)))\exp(-iHT) \\ &\times \exp(iH_0 t_1) - \exp(iH_0(t_2-t_1))\big|0\rangle = \int\limits_{0}^{\infty} dt_1 \int\limits_{0}^{\infty} dt_2 \exp(-i(t_1+t_2+T)) \\ &\times \int d^2\varrho_1 \int d^2\varrho_2 K_0(0,\boldsymbol{\varrho}_1,t_1)\left[K_c(\boldsymbol{\varrho}_1,\boldsymbol{\varrho}_2,T) - K_0(\boldsymbol{\varrho}_1,\boldsymbol{\varrho}_2,T)\right]K_0(\boldsymbol{\varrho}_2,0,t_2). \end{aligned} \tag{6.2}$$

Here a calculation of integrals over $\boldsymbol{\varrho}_1$ and $\boldsymbol{\varrho}_2$ may be performed e.g. in a such way:

- an integral over relative angle between $\boldsymbol{\varrho}_1$ and $\boldsymbol{\varrho}_2$ gives $J_0(\beta\varrho_1\varrho_2)$ where $J_0(x)$ is the Bessel function, $\beta = \beta_c = \dfrac{\nu}{2\sinh\nu T}$ and $\beta = \beta_0 = \dfrac{1}{2T}$ for the first and second terms in the square brackets in the right-hand side of (6.2),
- the remaining integrals over ϱ_1 and ϱ_2 can be found in tables.

So, we have

$$\begin{aligned} I_4^{(1)} &= \frac{1}{4\pi i}\int\limits_{0}^{\infty} dt_1 \int\limits_{0}^{\infty} dt_2 \exp\left(-i(t_1+t_2+T)\right)\left[N(t_1,t_2) - \frac{1}{t_1+t_2+T}\right], \\ N(t_1,t_2) &= \frac{\nu}{(1+\nu^2 t_1 t_2)\sinh\nu T + \nu(t_1+t_2)\cosh\nu T}. \end{aligned} \tag{6.3}$$

For $\nu_0 \gg 1$ the contribution into integral the term with $N(t_1,t_2)$ is of the order of $1/\nu_0$ and this term may be neglected. With allowance for $T \ll 1$ we find

$$\begin{aligned} I_4^{(1)} &= -\frac{1}{4\pi i}\int\limits_{0}^{\infty} dt_1 \int\limits_{0}^{\infty} dt_2 \exp\left(-i(t_1+t_2)\right)\frac{1}{t_1+t_2} \\ &= -\frac{1}{8\pi i}\int\limits_{0}^{\infty}\frac{dx}{x}\mathrm{e}^{-ix}\int\limits_{-x}^{x} dy = -\frac{1}{4\pi i}\int\limits_{0}^{\infty}\mathrm{e}^{-ix} = \frac{1}{4\pi}, \end{aligned} \tag{6.4}$$

where $x = t_1+t_2,\ y = t_1 - t_2$.

The contribution of the domain 4 into the term with r_2 ($I_4^{(2)}$ in (6.1)) contains two additional operators $\mathbf{p}$ (see (4.4)) which result additional factor $-\dfrac{\boldsymbol{\varrho}_1\boldsymbol{\varrho}_2}{4t_1t_2}$ in the integrand. Integration over the relative angle between $\boldsymbol{\varrho}_1$ and $\boldsymbol{\varrho}_2$ gives here $J_1(\beta\varrho_1\varrho_2)$ and subsequent evaluation of integrals is similar to those for (6.3). We find

$$I_4^{(2)} = -\frac{1}{4\pi}\int\limits_{0}^{\infty} dt_1 \int\limits_{0}^{\infty} dt_2 \exp\left(-i(t_1+t_2+T)\right)\left[N^2(t_1,t_2) - \frac{1}{(t_1+t_2+T)^2}\right]. \tag{6.5}$$

The contribution into the integral with $N^2(t_1,t_2)$ gives a domain
$t_1, t_2 \sim 1/\nu_0 \ll 1$. Since $T \ll 1$ as well, we can put an exponent in this integral equal to 1. So the integral is

$$\begin{aligned}\int\limits_0^\infty dt_1 \int\limits_0^\infty dt_2 N^2(t_1,t_2) &= \int\limits_0^\infty dx \int\limits_0^\infty dy \frac{1}{(axy+b(x+y)+a)^2} \\ &= 2\ln\frac{b}{a} = 2\ln\coth\nu T,\end{aligned} \tag{6.6}$$

where $x=\nu t_1,\ y=\nu t_2,\ a=\sinh\nu T,\ b=\cosh\nu T$. The second term in (6.5) is calculated as (see (6.4))

$$\begin{aligned}&\int\limits_0^\infty dt_1 \int\limits_0^\infty dt_2 \exp\left(-i(t_1+t_2+T)\right)\frac{1}{(t_1+t_2+T)^2} \simeq \int\limits_0^\infty dt e^{-it}\frac{t}{(t+T)^2} \\ &\simeq \int\limits_0^\infty dt e^{-it}\frac{t}{(t+T)} - 1 \simeq -C - \ln T - 1 + i\frac{\pi}{2},\end{aligned} \tag{6.7}$$

where $t=t_1+t_2$. Putting together (6.6) and (6.7) we have

$$I_4^{(2)} = 2\ln\tanh\nu T - C - \ln T - 1 + i\frac{\pi}{2}. \tag{6.8}$$

For computation of $I_1^{(2)} = I_3^{(2)}$ we will use Eq.(4.8)

$$\begin{aligned}I_1^{(2)} = \frac{\nu^2}{4\pi}\int\limits_0^\infty dt_1 \int\limits_0^T dt_2 \exp\left(-i(t_1+t_2)\right)\Bigg[&\frac{1}{(\nu t_1+\nu t_2)^2} \\ &-\frac{1}{(\sinh\nu t_2+\nu t_1\cosh\nu t_2)^2}\Bigg]\end{aligned} \tag{6.9}$$

Integrating by parts over t_1 with regard for $\exp(-it_2)\simeq 1$ we have

$$\begin{aligned}I_1^{(2)} &\simeq \frac{\nu}{4\pi}\int\limits_0^T dt_2\left[\frac{1}{\nu t_2} - \frac{1}{\cosh\nu t_2\sinh\nu t_2}\right] \\ &+\frac{\nu}{4\pi i}\int\limits_0^\infty dt_1\int\limits_0^T dt_2\exp(-it_1)\left[\frac{1}{\nu(t_1+t_2)} - \frac{1}{\cosh\nu t_2(\sinh\nu t_2+\nu t_1\cosh\nu t_2)}\right].\end{aligned} \tag{6.10}$$

The second of these integrals is proportional to T and can be neglected. The first integral gives

$$I_1^{(2)} = I_3^{(2)} = \frac{1}{4\pi}\ln\frac{\nu T}{\tanh\nu T}. \tag{6.11}$$

For a calculation of $I_2^{(2)}$ we use formulae (4.5) and (4.10)

$$\begin{aligned}I_2^{(2)} &\simeq \frac{\nu^2}{4\pi}\int\limits_0^T dt(t-T)\left[\frac{1}{\sinh^2\nu t} - \frac{1}{(\nu t)^2}\right] \\ &= \frac{\nu}{4\pi}\int\limits_0^T dt\left[\coth\nu t - \frac{1}{\nu t}\right] = \frac{1}{4\pi}\ln\frac{\sinh\nu T}{\nu T}.\end{aligned} \tag{6.12}$$

Combining all the contributions of four domains we obtain finally

$$\frac{dw}{d\omega} = \sum_{n=1}^4 \frac{4\alpha}{\omega}\mathrm{Re}\left(r_1 I_n^{(1)} + r_2 I_n^{(2)}\right) = \frac{\alpha}{\pi\omega}\mathrm{Re}\left[r_1 + \left(\ln(\nu\sinh\nu T) - 1 - C\right)r_2\right]. \tag{6.13}$$

In the used units ($T = al/2$) the formation length (2.1) is (see also (2.5) and (2.15))

$$t_c = \frac{al_c}{2} = \frac{1}{\zeta_c} = \varrho_c^2 = \frac{1}{\nu_0+1}. \tag{6.14}$$

In the case of thick target ($T \gg t_c,\ \nu_0 T \gg 1$) we have from (6.13)

$$\frac{dw}{d\omega} \simeq \frac{\alpha}{\pi\omega}\left[r_1 + (\ln\nu_0 - 1 - C - \ln 2)\, r_2\right] + \frac{\alpha T}{\pi\omega} r_2 \frac{\nu_0}{\sqrt{2}}, \quad \frac{\alpha T}{\pi\omega} = \frac{\alpha a l}{2\pi\omega} = \frac{\alpha}{2\pi\gamma^2}\frac{\varepsilon}{\varepsilon'} l. \tag{6.15}$$

This formula gives the probability of radiation at $\nu_0 \gg 1$ (see (2.28), (2.44)) where the contribution of boundary photons (4.14) is included.

In the case of thin target $\nu_0 T \ll 1$ but when $\nu_0^2 T \gg 1$ we have from (6.13) the probability (5.15) without term $\propto 1/L_t$. So we have ($\nu_0^2 T = 4k$)

$$\frac{dw}{d\omega} \simeq \frac{\alpha}{\pi\omega}\left[r_1 + \left(\ln(\nu_0^2 T) - 1 - C\right) r_2\right]. \tag{6.16}$$

Note, that when the value of the parameter ν_0 is not very large, the accuracy of the formulae (6.15) and (6.16) may be insufficient. In this case one have to compute the next terms of the expansion, as it was done in Sections 4 and 5 (see (4.14) and (5.15)). The same is true for (6.13). A detailed analysis of the probability of radiation in the targets of an intermediate thickness will be carry out elsewhere.

7 A qualitative behavior of the spectral intensity of radiation

We consider the spectral intensity of radiation for the energy of the initial electrons when the LPM suppression of the intensity of radiation takes place for relatively soft energies of photons: $\omega \leq \omega_c \ll \varepsilon$:

$$\nu_0(\omega_c) = 1, \quad \omega_c = \frac{16\pi Z^2\alpha^2}{m^2}\gamma^2 n \ln\frac{a_{s2}}{\lambda_c}, \tag{7.1}$$

see Eqs.(2.9), (2.14), (2.15) and (2.37). This situation corresponds to the experimental conditions.

A ratio of a thickness of a target and the formation length of radiation (2.1) is an important characteristics of the process. If we take into account the multiple scattering and the polarization of a medium then the formation length (3.3) has the form

$$l_f = \frac{2\gamma^2}{\omega}\left[1 + \gamma^2\vartheta_c^2 + \left(\frac{\gamma\omega_0}{\omega}\right)^2\right]^{-1}, \tag{7.2}$$

this ratio may be written as

$$\begin{aligned} \beta(\omega) &= T(\nu_0 + \kappa) \simeq T_c\left[\frac{\omega}{\omega_c} + \sqrt{\frac{\omega}{\omega_c}} + \frac{\omega_p^2}{\omega\omega_c}\right], \\ T &= \frac{l\omega}{2\gamma^2}, \quad \omega_p = \omega_0\gamma, \quad T_c \equiv T(\omega_c) \simeq \frac{2\pi}{\alpha}\frac{l}{L_{rad}}, \end{aligned} \tag{7.3}$$

where we put that $\nu_0 \simeq \sqrt{\frac{\omega_c}{\omega}}$. Below we assume that $\omega_c \gg \omega_p$ which is true under the experimental conditions.

If $\beta(\omega_c) = 2T_c \ll 1$ then at $\omega = \omega_c$ a target is thin and the Bethe-Heitler spectrum of radiation, which is valid at $\omega \gg \omega_c$ $\left(\frac{dI(\omega)}{d\omega} = \text{const}\right)$ will be also valid at $\omega \leq \omega_c$ in accordance with Eqs.(5.7) and (5.8) since $4k = \nu_0^2 T = T_c \ll 1$. This behavior of the spectral curve will continue with ω decrease until photon energies where a contribution of the transition radiation become essential.

If $\beta(\omega_c) \gg 1$ ($T_c \gg 1$) then at $\omega \geq \omega_c$ a target is thick and one has the LPM suppression for $\omega \leq \omega_c$. There are two opportunities depending on the minimal value of the parameter β.

$$\beta_m \simeq \frac{3}{2}T_c\sqrt{\frac{\omega_1}{\omega_c}}, \quad \omega_1 = \omega_p\left(\frac{4\omega_p}{\omega_c}\right)^{1/3}, \quad \beta_m \simeq 2T_c\left(\frac{\omega_p}{\omega_c}\right)^{2/3}. \tag{7.4}$$

If $\beta_m \ll 1$ then for photon energies $\omega > \omega_1$ it will be ω_2 such that

$$\beta(\omega_2) = 1, \quad \omega_2 \simeq \frac{\omega_c}{T_c^2} \tag{7.5}$$

and for $\omega < \omega_2$ the thickness of a target becomes smaller than the formation length of radiation so that for $\omega \ll \omega_2$ the spectral distribution of the radiation intensity is described by formulae of Section 5. Under these conditions for $4k = \nu_0^2 T = T_c \gg 1$ the spectral curve has a plateau

$$\frac{dI}{d\omega} = \frac{2\alpha J}{\pi} = \text{const} \tag{7.6}$$

in accordance with (5.13). Such behavior of the spectral curve (first discussed in [21]) will continue until photon energies where one has to take into account the polarization of a medium and connected with it a contribution of the transition radiation.

At $\beta_m \gg 1$ a target remains thick for all photon energies and radiation is described by formulae of Sections 2 and 3. In this case at $\omega \ll \omega_c$ ($\nu_0 \gg 1$) and $\omega \gg (\omega_p/\omega_c)^{1/3}\omega_p$ ($\nu_0 \gg \kappa$) the spectral intensity of radiation formed inside a target is given by Exp.(2.40) and (2.44) and the contribution of the boundary photons is given by (4.14).

Since a contribution into the spectral intensity of radiation from a passage of an electron inside of a target ($\propto T$) is diminishing and a contribution of the boundary photons is increasing with ω decrease, the spectral curve has a minimum at $\omega = \omega_m$. The value of ω_m may be estimated from equation (see (2.44) and (4.14))

$$\begin{aligned} &\frac{d}{d\omega}\left(\frac{\nu_0 T}{\sqrt{2}} + \ln\nu_0 + \frac{\pi^2\sqrt{2}}{24}\frac{\kappa}{\nu_0}\right) = 0, \quad \frac{\nu_0 T}{\sqrt{2}} \simeq 1 + \frac{\pi^2(\kappa-1)}{4\sqrt{2}\nu_0}, \\ &T_c \simeq \left(\frac{2\omega_c}{\omega_p}\right)^{1/2}\sqrt{x} + \frac{\pi^2}{4}x^2, \quad x = \frac{\omega_p}{\omega}. \end{aligned} \tag{7.7}$$

When a value of T_c is high enough, the solution of Eq.(7.7) doesn't satisfy the condition $\nu_0 \gg \kappa$ and in this case the equation (7.7) ceases to be valid. For determination of ω_m in this case we use the behavior of the spectral intensity of radiation at $\kappa \gg \nu_0$. In this case a contribution into radiation from inside passage of a target is described by (3.13) whilst the radiation of the boundary photons reduces to the transition radiation and its contribution is given by (4.11). Leaving the dominant terms ($\nu_0^2 T$ is ω-independent) we have

$$\frac{d}{d\omega}\left(\frac{\nu_0^2 T}{3\kappa} + \ln\kappa\right) = 0, \quad \frac{\nu_0^2 T}{3\kappa} = 1, \quad \kappa_m = \frac{T_c}{3}, \quad \omega_m \simeq \sqrt{\frac{3}{T_c}}\omega_p. \tag{7.8}$$

Since the value $\pi^2/12 \simeq 0.8$ is of the order of unity, the solution of (7.7) at $\kappa_m \gg \nu_0$ differs only slightly from ω_m. Because of this, if the condition $2T_c(\omega_p/\omega_c)^{2/3} \gg 1$ is fulfilled, the position of the minimum is defined by Eq.(7.7).

8 Discussion and conclusions

Now we consider the experimental data [12]-[14] from a point of view of the above analysis. It is shown that the mechanism of radiation depends strongly on the thickness of a target. So, we start with an estimate of thickness of used targets in terms of the formation length of radiation. From Eq.(7.3) we have that

$$T_c = \frac{2\pi l}{\alpha L_{rad}} \geq 20 \quad \text{at} \quad \frac{l}{L_{rad}} \geq 2\%.$$

The minimum value of the ratio of a thickness of a target to the formation length of radiation is given by Eq.(7.4) ($\beta_m \simeq 2T_c(\omega_p/\omega_c)^{2/3}$). For defined value of T_c this ratio is least of all for the heavy elements. Indeed, the value of $\omega_p = \omega_0\gamma$ depends weakly on nucleus charge Z ($\omega_0 = 30 \div 80\ eV$), while $\omega_c = \dfrac{4\pi\gamma^2}{\alpha L_{rad}} \propto Z^2$. Furthermore, the ratio ω_p/ω_c decreases with energy increase. Thus, among all targets with thickness $l \geq 2\% L_{rad}$ the minimal value of β_m is attained for the heavy elements (W, Au, U) at the initial electron energy $\varepsilon = 25\ GeV$. In this case one has $\omega_c \simeq 250\ MeV$, $\quad \omega_p \simeq 4\ MeV$, $\quad \beta_m \geq 2.5$. Since the parameter T_c is energy independent and the ratio $\omega_p/\omega_c \propto 1/\varepsilon$, the minimal value $\beta_m \geq 5$ is attained at the initial electron energy $\varepsilon = 8\ GeV$ for all targets with thickness $l \geq 2\% L_{rad}$ which can be considered as thick targets.

As an example of obtained results we calculated the spectrum of the intensity of radiation in the tungsten target with thickness $l = 2\% L_{rad}$ at the initial electron energy $\varepsilon = 8\ GeV$ and $\varepsilon = 25\ GeV$. The characteristic parameters of the radiation process for this case are given in the Table. We calculated the main (Migdal) term (Eq.(2.28)), the correction term (Eqs.(2.33),(2.41)) taking into account an influence of the polarization of a medium according to (Eq.(3.11)), as well as Coulomb corrections entering the parameters ν_0 (Eq.(2.10)) and $L(\varrho_c)$ (Eq.(2.36)). The contribution of an inelastic scattering of a projectile on atomic electrons (quite small for the heavy elements) is not included although this could be done using Eq.(2.46). We calculated also the contribution of the boundary photons Eq.(4.12). Here in the soft part of the spectrum $\omega < \omega_d (\omega_d \simeq 2\ MeV$ for $\varepsilon = 25\ GeV)$ the transition radiation term (4.11) dominates in (4.12), whilst in the harder part of the boundary photon spectrum $\omega > \omega_d$ the terms depending on both the multiple scattering and the polarization of a medium (4.9) and (4.10) give the main contribution; for $\varepsilon = 8\ GeV$ we have $\omega_d \simeq 700\ KeV$. It is seen that we have for the boundary photons spectrum a smooth curve which eliminate difficulties mentioned in [14].

Table: Characteristic parameters of the radiation process in tungsten with the thickness $l = 2\% L_{rad}$

ε (GeV)	ω_c (MeV)	ω_p (MeV)	T_c	ω_1 (MeV)	β_m	ω_m (MeV)
25	228	3.93	21.25	1.6	2.7	2
8	23.35	1.26	21.25	0.76	5.7	0.5

All these results presented separately in Fig.2 as well as their sum (curve 5). Note, that for energy $\varepsilon = 25\ GeV$ in the region of the minimum of the spectral curve 5 where the ratio of the target thickness to the formation length is minimal ($\beta_m \simeq 2.7$, see Table) it may be that the target is not thick enough to use the formulae for a thick target. For a comparison with experiment we extract some data from Fig.7 of [14]. The theoretical curve gives the spectral distribution of the intensity of radiation (in units $2\alpha/\pi$) without adjusting parameters. Data from [14] were recalculated according with procedure given in it. One can see that agreement between the experiment and theory is rather satisfactory but far from being perfect. However, one has to take into account that the theory of LPM effect in all previous papers had the logarithmic accuracy and did not contain Coulomb corrections. These shortcomings did not permit to pass to the Bethe-Heitler formula with acceptable accuracy and led to some difficulties in data processing. Both these shortcomings are overcome in the present paper. So, in our opinion, it is quite desirable to handle the experimental data using the formulae of this paper.

The measurements in [14] were made also using gold target with thickness $l = 0.7\% L_{rad}$. For this case one has $T_c \simeq 6$, $\quad \beta_m(25) = 0.7$, $\quad \beta_m(8) = 1.5$, so we have here a target of an intermediate thickness (see Section 6). We want to stress once more that for estimation of an effective thickness one have to use the formation length with regard for the multiple scattering and the polarization of a medium (see (3.3) and (7.2)). A detailed calculation for this case will be published elsewhere.

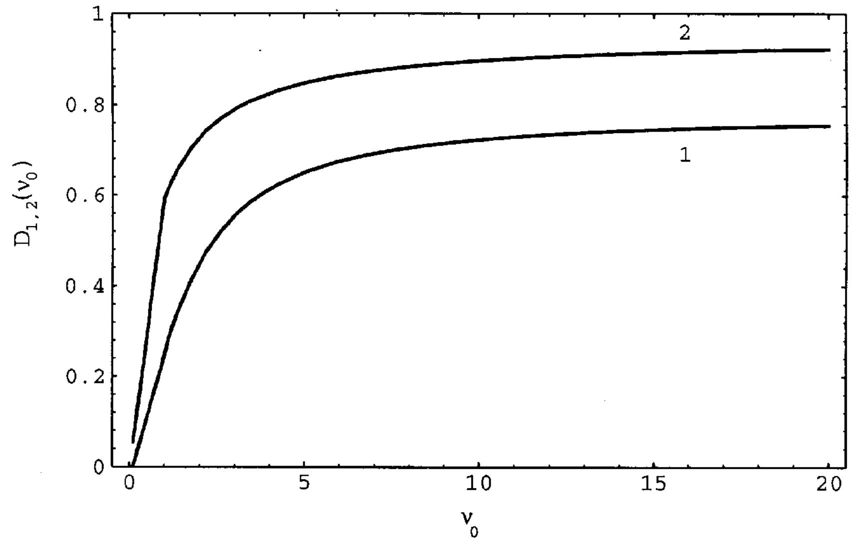


Figure 1: The functions $D_{1,2}(\nu_0)$ (Eq.(2.41)) vs parameter ν_0.

A Appendix

A potential $V(\varrho)$ with the Coulomb corrections

It is well known, that for heavy elements the Coulomb correction to the cross section of bremsstrahlung of high energy particles (correction to the Born approximation) is quite sizable, see e.g. Eq.(18.30) in [7]. The Coulomb correction (order of one for heavy elements) is subtracted from the "large" logarithm and if an accuracy of calculation goes beyond logarithmic one, one has to take into account this correction. For tungsten ($Z = 74$), gold ($Z = 79$) and uranium ($Z = 92$) in the case of complete screening the relative Coulomb corrections to the standard Bethe-Heitler cross section are respectively -7.5% , -8.3% and -10.7%.

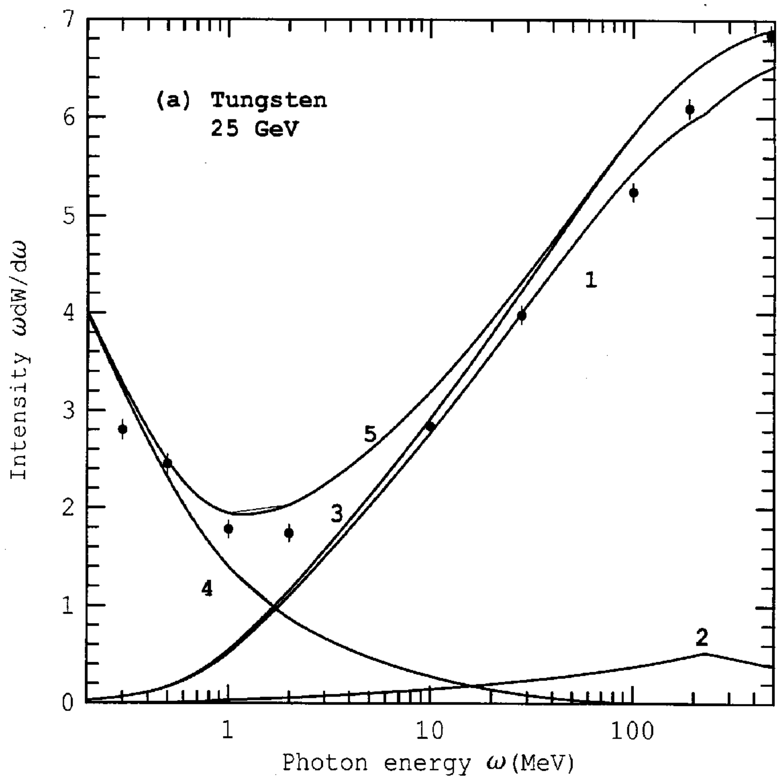
We consider the problem using eikonal approximation (see e.g.Appendix E in [7]). An amplitude $f(\mathbf{q})$ and a cross section of scattering in this approximation have the form:

$$f(\mathbf{q}) = \frac{1}{2\pi i}\int d^2\varrho \exp(-i\mathbf{q}\boldsymbol{\varrho})S(\boldsymbol{\varrho}), \quad S(\boldsymbol{\varrho}) = \exp\left(-i\chi(\boldsymbol{\varrho})\right) - 1,$$
$$\chi(\boldsymbol{\varrho}) = \int\limits_{-\infty}^{\infty} U(\boldsymbol{\varrho}, z)dz, \quad d\sigma(\mathbf{q}) = |f(\mathbf{q})|^2 d^2q, \tag{A.1}$$

where $(z, \boldsymbol{\varrho})$ are the longitudinal and transverse coordinates respectively, $U(\boldsymbol{\varrho}, z)$ is the potential. Repeating a derivation made in Section 2 (eqs. (2.3)-(2.9)) but with the cross section (A.1) we find for the potential $V(\boldsymbol{\varrho})$

$$V(\boldsymbol{\varrho}) = n\int \left(1 - \exp(i\mathbf{q}\boldsymbol{\varrho})\right)|f(\mathbf{q})|^2 d^2q$$
$$= n\int d^2x \left(S(\mathbf{x})S^*(\mathbf{x}) - S(\mathbf{x}+\boldsymbol{\varrho})S^*(\mathbf{x})\right). \tag{A.2}$$

Since the potential $V(\boldsymbol{\varrho})$ was calculated above in the Born approximation, we can calculate here the difference of the potentials calculated in the Born approximation $V_B(\boldsymbol{\varrho})$ and in the eikonal approximation



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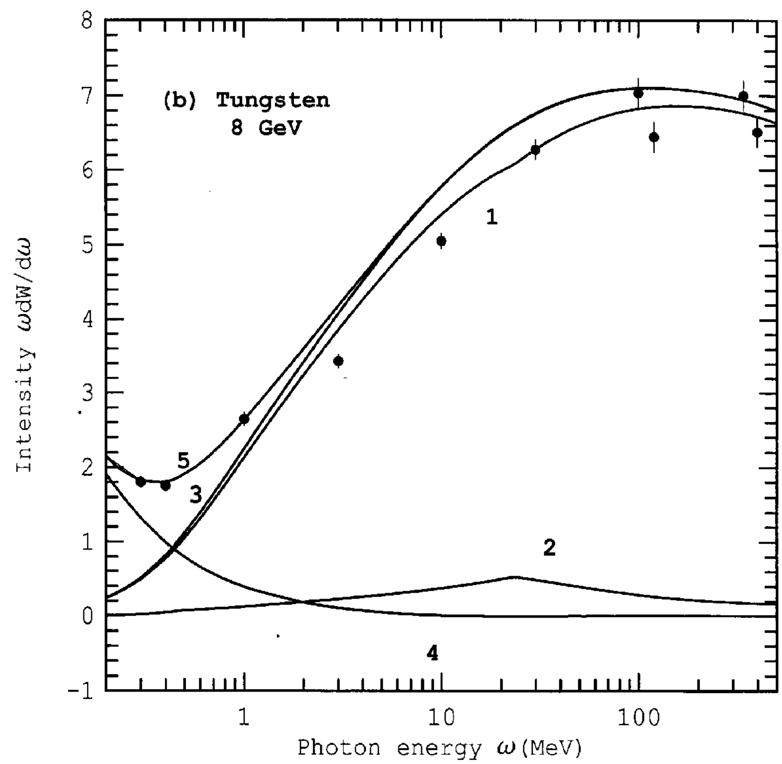

b

Figure 2: The intensity of radiation $\omega\dfrac{dW}{d\omega}$ in tungsten with thickness $l = 0.088\ mm$ in units $\dfrac{2\alpha}{\pi}$, ((a) is for the initial electrons energy $\varepsilon = 25\ GeV$ and (b) is for $\varepsilon = 8\ GeV$). The Coulomb corrections and the polarization of a medium are included:

– curve 1 is the contribution of the main term (2.28);

– curve 2 is the correction (2.33), (2.41);

– curve 3 is the sum of the previous contributions;

– curve 4 is the contribution of the boundary photons (4.12);

– curve 5 is the total prediction for the intensity of radiation.

$V(\varrho)$

$$\Delta V(\varrho) = V_B(\varrho) - V(\varrho) = n \int d^2x \Big\{ \exp\left[i\chi(\varrho + \mathbf{x}) - i\chi(x)\right] - 1$$
$$+ \tfrac{1}{2}\left[\chi(\varrho + \mathbf{x}) - \chi(x)\right]^2 \Big\};$$
$$\chi(x) = \int\limits_{-\infty}^{\infty} dz \frac{Z\alpha}{r} \exp\left(-\frac{r}{a_s}\right) = 2Z\alpha K_0\left(\frac{x}{a_s}\right), \quad r = \sqrt{z^2 + x^2}, \tag{A.3}$$

where $K_0(z)$ is the modified Bessel function. Because the eikonal phase enters (A.3) only in the combination $\chi(\varrho + \mathbf{x}) - \chi(x)$ in the interesting for us region $x \sim \varrho$. Since $K_0\left(\dfrac{x}{a_s}\right)$ is large only if $x/a_s \ll 1$ it is evident that main contribution into integral (A.3) gives the region $x \sim \varrho \ll a_s$. In this region one has

$$\chi(\varrho + \mathbf{x}) - \chi(x) = \xi \ln \frac{x^2}{(\varrho + \mathbf{x})^2}, \quad \xi = Z\alpha. \tag{A.4}$$

So, in the expression for $\Delta V(\varrho)$ enters only one dimensional parameter $\varrho = \mathbf{l}\varrho$, where $\mathbf{l}$ is the unit vector. After substitution of variables $\mathbf{x} \to \varrho\mathbf{x}$ we have

$$\frac{\Delta V(\varrho)}{n} = 2\pi\varrho^2\xi^2 f(\xi),$$
$$f(\xi) = \frac{1}{2\pi\xi^2} \int d^2x \left[\left(\frac{(\mathbf{x}+\mathbf{l})^2}{x^2}\right)^{i\xi} - 1 + \frac{\xi^2}{2}\ln^2\frac{(\mathbf{x}+\mathbf{l})^2}{x^2}\right]. \tag{A.5}$$

Changing variables $\mathbf{y} = \dfrac{\mathbf{x}}{x^2}$ and then $\mathbf{z} = \mathbf{y} + \mathbf{l}$ we have

$$f(\xi) = \frac{1}{2\pi\xi^2} \int \frac{d^2z}{(\mathbf{z}-\mathbf{l})^4} \left[z^{2i\xi} - 1 + \frac{\xi^2}{2}\ln^2 z^2\right]. \tag{A.6}$$

Integration over azimuthal angle gives

$$\int\limits_0^{2\pi} \frac{d\phi}{(z^2 - 2z\cos\phi + 1)^2} = \frac{2\pi(1+z^2)}{|z^2-1|^3}. \tag{A.7}$$

Changing the variable $z^2 = u$, splitting the integration interval into two parts: $(0,1)$ and $(1,\infty)$ changing in the second interval $v = 1/u$ we obtain

$$f(\xi) = \frac{1}{\xi^2}\mathrm{Re}\int\limits_0^1 \frac{du(1+u)}{(1-u)^3}\left(u^{i\xi} - 1 + \frac{\xi^2}{2}\ln^2 u\right) \tag{A.8}$$

Integrating by parts and changing once more variable $u = e^{-y}$ we find

$$f(\xi) = \frac{1}{\xi^2}\mathrm{ReRe}\int\limits_0^{\infty} \frac{e^{-y}dy}{(1-e^{-y})^2}\left[-i\xi e^{-i\xi y} + \xi^2 y\right]. \tag{A.9}$$

Integrating once more by parts and using the standard (Gauss) representation of the Euler ψ-function we have finally

$$\frac{\Delta V(\varrho)}{n} = 2\pi\varrho^2(Z\alpha)^2 f(Z\alpha),$$
$$f(\xi) = \mathrm{Re}\left[\psi(1+i\xi) - \psi(1)\right] = \xi^2\sum_{n=1}^{\infty}\frac{1}{n(n^2+\xi^2)}. \tag{A.10}$$

The obtained function $f(\xi)$ is the known Coulomb correction to the Bethe-Heitler cross section of bremsstrahlung, see e.g. [7], Sections 17,18.

B Appendix

An allowance for a form factor of a nucleus

When $\varrho_c \ll R_n$ (see (2.15), (2.16)) one cannot consider the potential of a nucleus as a potential of a point charge. A contribution into the multiple scattering gives a momentum transfer $q \leq 1/R_n$. Because of the same reason the phase $\mathbf{q}\boldsymbol{\varrho}$ in expression (A.2) for the potential $V(\boldsymbol{\varrho})$ is small $q\varrho \leq \varrho_c/R_n \ll 1$ and one can expand the potential. As a result we obtain

$$
\begin{aligned}
V(\varrho) &= \frac{n\varrho^2}{4}\int |\mathbf{q}f(q)|^2 d^2q = \frac{n\varrho^2}{4}\int |\boldsymbol{\nabla} S(\mathbf{x})|^2 d^2x \\
&= \frac{n\varrho^2}{4}\int \left(\boldsymbol{\nabla}\chi(\mathbf{x})\right)^2 d^2x = \frac{n\varrho^2}{4}\int \left(\mathbf{q}(\mathbf{x})\right)^2 d^2x,
\end{aligned}
\tag{B.1}
$$

where $\mathbf{q}(\mathbf{x})$ is the classical momentum transfer on a straight-line trajectory with an impact parameter $\mathbf{x}$. As one can see from (B.1), the mean square of the momentum transfer is the same the in eikonal approximation, in the Born approximation and in the classical theory. The Coulomb correction in this case vanishes.

Considering a nucleus as an uniformly charged sphere with the radius R_n, we have

$$
\begin{aligned}
&\mathbf{q}(\boldsymbol{\varrho}) = \frac{2\xi\boldsymbol{\varrho}}{\varrho^2}\left[\frac{\varrho}{a_s}K_1\left(\frac{\varrho}{a_s}\right)\vartheta\left(\varrho - R_n\right) + \varphi\left(\frac{\varrho^2}{R_n^2}\right)\vartheta\left(R_n - \varrho\right)\right], \\
&\varphi(x) = 1 - \sqrt{1-x} + x\sqrt{1-x}.
\end{aligned}
\tag{B.2}
$$

Substituting the expression obtained into (B.1) we find the potential $V(\varrho)$ under conditions considered

$$
\begin{aligned}
&\int q^2(\varrho)d^2\varrho = 4\pi\xi^2\left[\frac{2}{a_s^2}\int\limits_{R_n}^{\infty} K_1^2\left(\frac{\varrho}{a_s}\right)\varrho d\varrho + \int\limits_0^1 \frac{dx}{x}\varphi^2(x)\right] \\
&= 4\pi\xi^2\left[\left(\ln\left(\frac{2a_s}{R_n}\right)^2 - 1 - 2C\right) + \left(\frac{7}{2} - 4\ln 2\right)\right] \\
&= 4\pi Z^2\alpha^2\left[\ln\frac{a_s^2}{R_n^2} + \frac{5}{2} - 2(C + \ln 2)\right]; \\
&V(\varrho) = \pi\xi^2 n\varrho^2\left[\ln\left(\frac{a_s^2}{R_n^2}\right) - 0.0407\right].
\end{aligned}
\tag{B.3}
$$

If one uses standard representation of nuclear form factor (see e.g. [17])

$$
F(q) = \frac{1}{(1+q^2\varrho_0^2)^2}, \quad \varrho_0^2 = \frac{R_n^2}{6}, \quad R_n = 1.2\cdot 10^{-13}A^{1/3}cm,
\tag{B.4}
$$

then one obtains

$$
\int q^2(\varrho)d^2\varrho = 4\pi Z^2\alpha^2\left[\ln\frac{a_s^2}{R_n^2} + \ln 6 - 2\right] \simeq 4\pi Z^2\alpha^2\left[\ln\frac{a_s^2}{R_n^2} - 0.208\right].
\tag{B.5}
$$

Taking into account that $\ln\dfrac{a_s^2}{R_n^2} \simeq 20$ we see that the difference between different models of nucleus is less than 1 % .

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