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Abstract

Using the BRM theory developed recently by Fyodorov and Mirlin we calculate the density-density correlators for Banded Random Matrix of infinite size. Within the accuracy of $1/b^2$ (b is the matrix bandwidth) it appears to be the same in both cases of the orthogonal and unitary symmetry. Moreover, its form coincides exactly with the formula obtained long ago by Gogolin for the electron density-density correlator in strictly 1D disordered metals.

In addition to the "fixed energy" density-density correlator considered in the solid state physics we calculate also the "time averaged" one, which has different properties at small separations. Our predictions are compared with the existing numerical data.

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1 Introduction

The Band Random Matrix(BRM) model is a good basis[1] for understanding basic features of quantum chaotic systems. It looks even more attractive in view of its analytical solution obtained by Fyodorov and Mirlin [5] in the framework of the Efetov's supersymmetry approach [6]. Being inspired by numerous earlier attempts [7] to understand scaling properties of the BRM model, the solution is proved to be an adequate tool to address the problem.

Recently, further extensive numerical studies of the RBM model for the real symmetric matrices (we refer to this case as that of orthogonal ensemble) has been done in the work [3]. Among many of interesting results concerning the scaling properties of the BRM model, very detailed data were presented for the *time-averaged* density-density correlator, known as the "steady state distribution"

$$\mathcal{K}_t(x_1,x_2) = \overline{\lim_{T\to\infty}\frac{1}{T}\int_0^T dt\,|\langle x_2,t|x_1,0\rangle|^2} = \overline{\sum_\alpha |\psi_\alpha^*(x_2)\,\psi_\alpha(x_1)|^2}, \qquad (1)$$

where ψ_α are eigenfunctions of the Hamiltonian considered to be a member of BRM ensemble: $H\psi_\alpha = E_\alpha\psi_\alpha$ and E_α is the corresponding eigenvalue. The scaled coordinates enetering Eq.(1) are $x_{1,2} = n_{1,2}/b^2$, with $n_{1,2}$ being labels of the components of BRM eigenvectors and b being the band widths. The bar stands for the averaging over the random matrix ensemble.

In [3] the data for the correlator (1) has been fitted by the theoretical expression obtained long time ago by Gogolin [8] for strictly 1D Anderson model. The Gogolin's expression contains the localisation length l as a parameter. A reasonable agreement has been found for the length parameter value $l_m \approx 0.29b^2$. This value is rather close to the localization length $l_\infty|_{(E=0)} = \frac{1}{3}b^2$ expected for the *orthogonal* BRM ensemble with the matrix size $N \gg b^2 \gg 1$ [2]

[1] See, e.g., [1, 2, 3] as well as a recent very extensive review [4], and references therein.

[2] Note, that in Ref.[3] it was erroneously concluded that that l_m differs substantially from l_∞. This is however a consequence of incorrect use of the localization length l_∞ defined for the *unitary* (see Eq.(4),(5) in Ref.[3]) instead of the *orthogonal* BRM ensemble. The localisation length l_∞ for the unitary class is known to be twice as large in comparison with that for the orthogonal class.)

However, the fit done in [3] is in fact not very legitimate. The reason is that the Gogolin expression is not relevant for the correlator (1) and represents another quantity: a *"fixed energy"* correlator defined as

$$\mathcal{K}_E(x_1, x_2; E) = \overline{\sum_\alpha |\psi^*_\alpha(x_2, E_\alpha)\psi_\alpha(x_1, E_\alpha)|^2 \delta(E - E_\alpha)}. \tag{2}$$

It is easy to see, that time-averaged correlator can be obtained from the "fixed energy" one by integration over the energy. Since the localization length parameter depends essentialy on the energy, the integration can modify the correlator form. Indeed, as we show below the Gogolin correlator with energy dependent localization parameter integrated over the energy is a better fit for the data.

Actualy, the best way is to fit the data by theoretical predictions following from the BRM model itself, rather than to use Gogolin's formula derived formally for diferent case. Unfortunately, the relevant quantity was not calculated up to now. In particular, in [9] the correlator $\mathcal{K}_E(x_1, x_2; E)$ has been calculated at points $n(x_1) = 0$, $n(x_2) = N$ (N is matrix size), while to compare with data [3] one needs such a correlator for finite separations $x_2 - x_1$ but infinite matrix size $N \to \infty$.

This work is intended to fill this gap. We calculate the density-density correlator for infinite banded random matrices of two symmetry classes: the *orthogonal* ensemble, studied in [3], and the *unitary* one, consisting of complex Hermitian banded matrices. The paper is organized as follows: after a brief overview of mapping of the BRM model onto the supersymmetrical σ-model in Sec.2, we consider step by step the transfer matrix solution of the σ-model in Sec.3 and calculate the *fixed energy* correlator in Sec.4 for both symmetry cases simultaneously. As a result, we find that within an accuracy of $O(1/b^2)$ both correlators coincide in their functional form. At the same time, the localization length is as expected twice as smaller in the orthogonal ensemble as compared to the unitary one. Integrating in Sec.5 the "fixed energy" correlator over the energy we complete the paper with BRM predictions for the time averaged correlator, the "steady state distribution".

2 Basic definitions of the BRM problem and its mapping onto the supersymmetric nonlinear σ-model.

Let us consider the ensemble of $N \times N$ ($N \gg 1$) banded random matrices H, whose elements are independently distributed according to the Gaussian law with zero mean value and variances

$$\overline{|H_{ij}|^2} = J_{ij} \equiv f(|i - j|), \tag{3}$$

The band profile function $f(r)$ vanishes exponentially or faster outside the band $|i - j| \leq b$, $b \gg 1$. In this case, as it was known from [5] predictions are insensitive to the particular band profile form.

Below we consider simultaneously two symmetry classes of matrices H: the *orthogonal* ensemble of real symmeric matrices, and the *unitary* ensemble of hermitian ones.

The former corresponds to systems which are time-reversal invariant , while the latter corresponds to broken invariance.

One can get a solution of the BRM problem by mapping it onto supersymmetric σ-model studied in many detailes in earlier Efetov's works [10, 6]. Let us summarize the results [5, 2] of such a mapping for both symmetry classes. The corresponding σ-model action reads:

$$S\{Q\} = -\frac{\gamma}{2}\sum_i \mathrm{Str} Q(i)Q(i+1) + i\epsilon\sum_i \mathrm{Str} Q(i)\Lambda, \tag{4}$$

in terms of which the density-density correlator (2) is expressed as:

$$\mathcal{K}(n_1,n_2;E) = -\lim_{\epsilon\to 0}\epsilon\cdot\rho C_0\int\prod_i d\mu[Q(i)]\ (Q^{11}_{bb,11}(n_1)Q^{22}_{bb,11}(n_2)$$

$$+C_\times\delta_{n_1,n_2}Q^{12}_{bb,11}(n_1)Q^{21}_{bb,11}(n_1))\exp(-S\{Q\}). \tag{5}$$

where the σ-model supermatrix field Q satisfyes a constraint: $Q^2 = 1$. With use the Efetov's notations the supermatrix Q consists of four supermatrix blocks

$$Q = \begin{pmatrix} Q^{11} & Q^{12} \\ Q^{21} & Q^{22} \end{pmatrix}, \tag{6}$$

where upper indices 1 and 2 correspond to the "retarded" and "advanced" components of the original RBM model. The bottom indices *bb* in eq.(5) denote boson-boson components inside the superblocks. For more detailed explanation see, e.g. reviews [11, 2].

The difference between the two symmetry classes is reduced to symmetry-specific definitions of the model parameters ρ, γ, C_0, $C_\times$ [3], as well as of the supermatrices Λ and Q.

In the *orthogonal* ensemble the 8×8 supermatrices Q belong to the coset space UOSP(2,2/2,2)/ UOSP(2/2)$\times$UOSP(2/2). Their explicit parametrization, including invariant measure $d\mu[Q]$ can be found, e.g. in [11, 12]. The diagonal matrix Λ is $\Lambda = \mathrm{diag}(1,1,1,1,-1,-1,-1,-1)$. The density of states ρ obeys the semicircle law

$$\rho = \frac{1}{\pi J_0}(E^2_{max} - E^2)^{1/2}, \qquad |E| \le E_{max} = (2J_0)^{1/2}, \tag{7}$$

and the "length scale" parameter γ is [13]

$$\gamma = \frac{(\pi\rho J_0)^2}{4}B = \frac{1}{4}B(2J_0 - E^2), \tag{8}$$

where the band profile parameters are defined as

$$J_0 = \sum_{r=-\infty}^{\infty} a(r), \qquad B = \sum_{r=-\infty}^{\infty} a(r)r^2/(J_0)^2. \tag{9}$$

The correlator parameters are $C_0 = 2$ and $C_\times = 2$, correspondingly.

[3] Note, that the factor $C_\times$ originates from the number of cross-pairing of components $1 \leftrightarrow 2$. Since this contribution gives rise at $n_1 = n_2$ only, it is of main importance in inverse participation ratio (IPR).

In the *unitary* ensemble the 4×4 supermatrices Q belong to the coset space U(1,1/2)/U(1/1)×U(1/1), and their explicit parametrization can be found, e.g. in [14, 2]. In this case the matrix Λ is $\Lambda = \mathrm{diag}(1, 1, -1, -1)$. The density of states ρ as well as the parameter γ are given by [2]:

$$\rho = \frac{1}{2\pi J_0}(E_{max}^2 - E^2)^{1/2}, \qquad |E| \leq E_{max} = (4J_0)^{1/2}, \tag{10}$$

$$\gamma = (\pi \rho J_0)^2 B = \frac{1}{4} B(4J_0 - E^2), \tag{11}$$

and differ from those in the orthogonal ensemble. It is easy to see, that in the unitary case the "length scale" γ (at $E = 0$) is twice as large and the correlator parameters are in this case: $C_0 = 1$, $C_\times = 1$.

At this level of consideration two symmetry classes differ mainly by the rescaling of the length scale γ and by explicit form of the supermatrices Q.

At the end of this section let us consider briefly the accuracy of the BRM model mapping onto the supersymmetrical σ-model. The only approximation done at this step is the integration over so-called "massive" modes performed by saddle-point method in Gaussian approximation(for more details see, e.g. [11, 2]). While the separation between the saddle points is of order (see, e.g. [11, 2])

$$\Delta \sim \pi \rho J_0 \sim J_0^{1/2}(E_{max}^2 - E^2)/E_{max}^2, \tag{12}$$

the variances of the massive modes are $\overline{(\delta P)^2} \propto J_0/b$ [2]. Hence, the ratio of the amplitude of the massive modes to the saddle points separation is of order of $O(b^{1/2})$:

$$r \sim \frac{\left(\overline{(\delta P)^2}\right)^{1/2}}{\Delta} \sim \frac{E_{max}^2}{b^{1/2}(E_{max}^2 - E^2)^{1/2}} \sim \left(\frac{E_{max}}{b(E_{max} - |E|)}\right)^{1/2}, \tag{13}$$

which, probably, defines the accuracy of the approximation.[4] Putting $r \sim 1$, one may also conclude, that semicircle law for the density of states is valid as long as $\Delta E = (E_{max} - |E|) \gg 1/b$.

3 Transfer matrix solution: from the discrete model to the continuous one.

Starting with the pioneer work [10], the transfer matrix technique looks as the most adequate tool to solve the problem. Forthcoming works [14, 12] make some detailes be more transparent. The continuous limit version [13] reduces the problem to solution of a partial differential equation. As we check below, this equation is the same for both the orthogonal and unitary ensemble, with accuracy $O(\gamma^{-1})$. The corresponding boundary

[4]Since the number of saddle points is, in fact, even, the cancellation of terms linear in r can occur. Then the accuracy will be better, of order of $O(b^{-1})$. More careful consideration is needed to arrive at definite conclusion.

conditions are also the same when calculating the density-density correlator, but can be symmetry dependent in more general case. To demonstrate this let us outline basic steps of the approach.

Following [2] consider the N-site one-dimensional chain of matrices $\{Q(k)\}$, with a nearest neighbour interaction given by eq.(4). For the sake of definiteness we assume that $n_1 < n_2$ in eq.(5). Having in mind that the integrations over different $Q(k)$, $k = 1, ..., (n_1 - 1)$ in eq.(5) can be performed subsequently one after another with the *same* kernel

$$L(Q, Q') = \exp\{\frac{\gamma}{2}\mathrm{Str}QQ' - i\epsilon\mathrm{Str}Q'\Lambda\}, \tag{14}$$

it is very convenient to consider the quantity $Y^{(1)}(Q; k)$ satisfying the recurrence relation

$$Y^{(1)}(Q; k) = \int d\mu(Q')L(Q, Q')Y^{(1)}(Q'; k-1). \tag{15}$$

with the initial condition $Y^{(1)}(Q; 0) \equiv 1$. In general, the integration over Q is quite nontrivial. A great simplification occurs in the limit $\epsilon \to 0$, which is exactly the case we are interested in (see the eq.(5)). As it was clearly demonstrated in [14, 12] with Efetov parametrization of matrices Q, after taking such a limit the function $Y^{(1)}(Q; k)$ depends only on one of its bosonic "eigenvalues" (say, λ_1) of the superblock Q^{11}, which takes values $\lambda_1 \sim \epsilon^{-1}$. Although the kernel (14) has a very extensive grassmannian part, in the limit $\epsilon \to 0$ only the term of the maximal grassmanian order survives due to the Parisi-Sourlas-Efetov-Wegner (PSEW) theorem[5]. This is the term of the eighth or fourth order in the orthogonal and unitary case, respectively. Perfoming explicit integration over all variables but λ_1 and using, that $Y^{(1)}(Q; k)$ depends only on $z = 2\epsilon\lambda_1$, one can rewrite the recurrence relation (15) in a more simple form:

$$Y^{(1)}(z; k) = \int_0^\infty dz' \; L_\gamma(\frac{z}{z'}) \exp(-z) Y^{(1)}(z'; k-1). \tag{16}$$

The explicit form of the kernel $L_\gamma(\frac{z}{z'})$ can be found, e.g. in [12] and [14] for the orthogonal and unitary case, respectively.

This recurrence relation can be used iteratively to construct the function $Y^{(1)}(z; k)$ up to the point $k = n_1$. At this point the integrand in eq.(5) has an extra factor $Q^{11}_{bb,11}$. This factor already depends on the whole set of the ordinary parameters of matrix Q and on a half subset of grassmann variables (the two α, α^* in the unitary case, and the four $\alpha_1, \alpha_1^*, \alpha_2, \alpha_2^*$ in the orthogonal case, in notations of papers [14, 12]) [6]. Omitting the detailes (some of them can be found in [12, 2]) we have found that in the limit $\epsilon \to 0$ the only nonvanishing contribution to the density-density correlator coming from the factor $Q^{11}_{bb,11}$ is the term of the maximal order in grassmann variables. It is of the kind $c(z, ...)G_{max}(\alpha)$, with

$$G_{max}(\alpha) \equiv \begin{cases} \alpha_1\alpha_1^*\alpha_2\alpha_2^*, & \text{orthogonal case,} \\ \alpha\alpha^*, & \text{unitary case,} \end{cases} \tag{17}$$

[5] See, for details [14, 15].

[6] Analogous factor $Q^{22}_{bb,11}$, with replacing $\beta \to \alpha$, arises at $k = n_2$.

and the ordinary coefficient $c(z,...)$, which (after averaging over the angle variables in the orthogonal case) is the same for both symmetry classes: $c(z,...) \to (-iz/(2\epsilon))$.

To proceed further, beginning from the point $k = n_1$ we start iterations with a new function $Y_G^{(2)}(Q;k,n_1)$ satisfying initial condition $Y_G^{(2)}(Q;k=n_1,n_1) = zY^{(1)}(z,n_1)G_{max}(\alpha)$. The subscript G denotes here, that this new function aquires a *nontrivial grassmann part.* As a result, one could suspect that the recurrence relation (16) is not valid any longer and one should return to the more complex relation (15). Fortunately, the closer inspection shows that to the leading order in $\epsilon \to 0$ the recurrence relation (15) acts on functions $Y_G^{(2)}(Q;k,n_1) = Y^{(2)}(z;k,n_1)\cdot G_{max}(\alpha)$ in such a way as if the grassmannian factor remains unchanged, while its *ordinary* coefficient $Y^{(2)}(z;k,n_1)$ is iterated by the relation (16) with *the same* kernel, as before.

The functions $Y^{(1)}(z;k)$ and $Y^{(2)}(z;k,n_1)$ allow one to calculate the correlator (5). Let us first consider the "direct pairing" contribution to eq.(5) coming from the term $Q^{11}_{bb,11}(k)Q^{22}_{bb,11}(l)$:

$$\mathcal{K}_d(n_1,n_2;E) = -\lim_{\epsilon\to 0}\cdot\epsilon\rho C_0 \int d\mu(Q)\; Y^{(1)}(z=2\epsilon\lambda_1;N-n_2)$$

$$\times Q^{22}_{bb,11} g_{max}(\alpha)\; (-\frac{i}{2\epsilon}) Y^{(2)}(z=2\epsilon\lambda_1;n_2,n_1)e^{-z}, \qquad (18)$$

where the only explicit integration over Q left is that done at the site $k = n_2$. Using the PSEW theorem, performing all the integrations[7] except for that over $z = 2\epsilon\lambda_1$, and finally taking the limit $\epsilon \to 0$ we come to the expression

$$\mathcal{K}_E^{(d)}(n_1,n_2;E) = \rho \int \frac{dz}{z^2}\; Y^{(1)}(z;N-n_2)\; z\; Y^{(2)}(z;n_2,n_1)e^{-z}. \qquad (19)$$

Note, that this expression is the same for both symmetry classes, but the functions $Y^{(1)}(z;n_2)$ and $Y^{(2)}(z;n_2,n_1)$ are calculated with kernels L_γ, which, generally speaking, are *dependent* on the symmetry class.

Another contribution to the correlator (5)comes from the cross-pairing term $Q^{12}_{bb,11}Q^{21}_{bb,11}$. It is even more easy to calculate. One can verify, that at the point $n_2 = n_1$ the term $Q^{12}_{bb,11}Q^{21}_{bb,11} \approx Q^{22}_{bb,11}Q^{11}_{bb,11}$ to the leading order in $\lambda_1 \sim 1/\epsilon \to \infty$. Therefore the full correlator can be written as

$$\mathcal{K}_E(n_1,n_2;E) = \mathcal{K}_E^{(d)}(n_1,n_2;E) + C_\times \delta_{n_1 n_2}\mathcal{K}_E^{(d)}(n_1,n_1;E). \qquad (20)$$

The final simplification arises if one considers the bandwidth b of the matrices to be large enough, so that the parameter $\gamma \sim b^2 \gg 1$. Let us note, that this condition was actually used in the previous section in order to map the original BRM problem onto the sypersymmetrical σ-model (see discussion in [2]).

First of all, the kernels $L_\gamma(z/z')$ obtained in [12] for the orthogonal case as well as that for the unitary case [14] coincide with each other within the accuracy $O(\gamma^{-1})$. Then one

[7]Note, that this integrations give the factor 1 and 2 in the orthogonal and unitary cases, respectively. With the factor C_0 this gives a factor 2 for the both symmetry cases.

can pass to the continuous limit by using that the kernel has a sharp peak. It amounts to replace the integral recurrence equation by the differential one [13, 2]:

$$\frac{\partial W^{(s)}(y,x)}{\partial \tau} = \hat{\mathbf{R}} W^{(s)}(y,x), \quad \hat{\mathbf{R}} \equiv \left\{ y^2 \frac{d^2}{dy^2} - y \right\}, \quad s = 1, 2; \tag{21}$$

where we introduced a continuous variable $x = k/2\gamma$, a scaled variable $y = 2\gamma z$, and the functions

$$W^{(1)}(y;x) = Y^{(1)}(z = y/2\gamma; k = 2\gamma x), \tag{22}$$

$$W^{(2)}(y;x,x_1) = 2\gamma \cdot Y^{(2)}(z = y/2\gamma; k = 2\gamma x, n_1 = 2\gamma x_1); \tag{23}$$

satisfying the initial conditions

$$W^{(1)}(y;0) = 1, \qquad W^{(2)}(y;x_1,x_1) = y W^{(1)}(y;x_1). \tag{24}$$

Let us stress, that the differential equation (21) as well as the initial conditions (24) are exactly the same for both symmetry classes[8].

The correlator (18) in continuous limit takes a form

$$\mathcal{K}_E^{(d)}(x_2,x_1;E) = \frac{\rho}{2\gamma} \int \frac{dy}{y^2}\, W^{(1)}(y;x_N - x_2)\; y\; W^{(2)}(y;x_2,x_1) e^{-y/2\gamma}, \tag{25}$$

where we introduced the dimensionless matrix size as $x_N = N/2\gamma$.

4 Calculation of the correlator $\mathcal{K}_E(x_1,x_2;E)$ for infinite banded matrices.

The way to obtain a solution of eq.(22) was in details considered in [13, 2]. For the function $W^{(1)}(y,x,x_0)$ one has

$$W^{(1)}(y;x) = 2y^{1/2} \Big\{ K_1(2y^{1/2}) + \frac{2}{\pi} \int_0^\infty d\nu_1 \frac{\nu_1}{1+\nu_1^2} \sinh(\pi\nu_1) K_{i\nu_1}(2y^{1/2})\, e^{-\frac{1+\nu_1^2}{4}x} \Big\}, \tag{26}$$

where $K_\alpha(t)$ is the MacDonald function. In this work we consider the case of the band matrices of the infinite size: $N/2\gamma \to \infty$. In contrast to [9, 2] the correlator arguments x_1, x_2 are taken at the finite separation $(x_2 - x_1)$, but both far from the ends $x = 0$, $x_N = N/2\gamma$. In this limit we have

$$W^{(1)}(y;x_1), W^{(1)}(y;x_N - x_2) \to 2y^{1/2} K_1(2y^{1/2}) \tag{27}$$

[8]Note, that in the course of calculations of other quantities, e.g. *the probability distribution* for the density-density correlator or inverse participation ratio, the corresponding initial conditions may be *symmetry dependent* [9].

since both $x_1 \to \infty$, and $(x_N - x_2) \to \infty$. Constructing the function $W^{(2)}(y, x, x_1)$ with the initial condition (24): $W^{(2)}(y, x_1, x_1) = y \cdot 2y^{1/2} K_1(2y^{1/2})$ we find

$$W^{(2)}(y; x_1, x_2) = \frac{y^{1/2}}{\pi^2} \int\limits_0^\infty d\nu_2 \, \nu_2 \sinh(\pi\nu_2)$$

$$\times e^{-\frac{(1+\nu_2^2)}{4}(x_2 - x_1)} K_{i\nu_2}(2y^{1/2}) \int\limits_0^\infty d\tilde{t}\, \tilde{t} K_{i\nu_2}(\tilde{t}). \tag{28}$$

Substituting the functions $W^{(1)}$ and $W^{(2)}$ into the correlator expression (25) one finally arrives at

$$\frac{1}{\rho}\mathcal{K}_E^{(d)}(x_2, x_1; E) = \frac{1}{2\gamma} \int\limits_0^\infty dy\, y^{-1} e^{-\frac{y}{2\gamma}} \cdot 2y^{1/2} K_1(2y^{1/2}) \times$$

$$\frac{y^{1/2}}{\pi^2} \int\limits_0^\infty d\nu_2 \, \nu_2 \sinh(\pi\nu_2)\, e^{-\frac{(1+\nu_2^2)}{4}(x_2 - x_1)} K_{i\nu_2}(2y^{1/2}) \int\limits_0^\infty d\tilde{t}\, \tilde{t} K_{i\nu_2}(\tilde{t}), \tag{29}$$

$$\simeq \frac{1}{2\pi^2\gamma} \int\limits_0^\infty d\nu_2 \, \nu_2 \sinh(\pi\nu_2)\, e^{-\frac{(1+\nu_2^2)}{4}(x_2 - x_1)} \left[\int\limits_0^\infty dt\, t K_1(t) K_{i\nu_2}(t) \right]^2$$

$$= \frac{\pi^2}{16 \cdot 2\gamma} \int\limits_0^\infty d\nu \, \nu \sinh(\pi\nu)\, e^{-\frac{(1+\nu^2)}{4}(x_2 - x_1)} \left(\frac{1 + \nu^2}{1 + \cosh \pi\nu} \right)^2 \tag{30}$$

In eq.(29) we omitted the factor $\exp(-y/2\gamma)$ since the integral over y converges at $y \sim 1$ but $2\gamma \sim b^2 \gg 1$. We see also, that in the infinite matrix size limit the correlator depends on the difference $x = x_2 - x_1$ only which means the restoration of the translational symmetry.

It is interesting, that the BRM model prediction for the *functional form* of the correlator $\mathcal{K}_E(x_2 - x_1; E)$ coinsides exactly with that obtained many years ago by Gogolin [8] for the strictly 1D electron system in disordered potential. However, the spatial scales in these systems are quite different. Namely, in the *strictly* 1D electron system it is equal to the mean free path defined by the Born amplitude of electron-impurity scattering, while in the *quasi*-1D BRM model it is governed by the localisation length $l_c = 2\gamma \propto b^2$, growing quadratically with matrix bandwidth.

5 RBM model predictions for time-average correlator $\mathcal{K}_t(x_1, x_2)$.

As we have already mentioned in the Introduction, the *time-averaged* correlator $\mathcal{K}_t(x_1, x_2)$ can be obtained from the "fixed energy" correlator $\mathcal{K}_E(x_1, x_2; E)$ by integrating the latter over the energy:

$$\mathcal{K}_t(x_1, x_2) = \int\limits_{-\infty}^{\infty} dE\, \mathcal{K}_E(x_1, x_2; E). \tag{31}$$

Substituting eq.(30) into eq.(31) we get

$$\mathcal{K}_t(x_2-x_1)=\int\limits_{-E_0}^{E_0} dE\,\frac{\pi^2\rho(E)}{16\cdot l_c(E)}.\int\limits_0^{\infty} d\nu\,\nu\sinh(\pi\nu)$$

$$\times\exp\left\{-\frac{(1+\nu^2)}{4}\frac{l_c(0)}{l_c(E)}(x_2-x_1)\right\}\left(\frac{1+\nu^2}{1+\cosh\pi\nu}\right)^2, \qquad (32)$$

with energy dependent localization length $l_c(E)=2\gamma(E)$. The scaled variables $x_{1,2}$ are related here to the discrete ones $n_{1,2}$ via the energy independent scale factor $l_c(0)$: $x_{1,2}=n_{1,2}/l_c(0)$.

For large $N,b\gg 1$ the density of states $\rho(E)$ in RBM model obeys the semicircle law:

$$\rho(E)=\rho(0)\cdot\left(1-\frac{E^2}{E_0^2}\right)^{1/2}=\frac{2}{\pi E_0}\left(1-\frac{E^2}{E_0^2}\right)^{1/2},\quad \int dE\,\rho(E)=1; \qquad (33)$$

and the localization length is, correspondingly,

$$l_c(E)=l_c(0)\cdot\left(1-\frac{E^2}{E_0^2}\right) \qquad (34)$$

with $E_0=2J_0$, $l_0(0)=J_0B/2\approx b^2/3$ or $E_0=4J_0$, $l_0(0)=BJ_0\approx 2b^2/3$ for the orthogonal or unitary cases, respectively (see eqs.(7)-(11)).

Let us make a remark about the limits of the integration. According to the note at the end of Sec.2, both the semicircle law for the density of states and the solution itself are valid as long as $(E_0-|E|)\gg\delta E\sim E_0/b$. Thus, extending the integration in the Eq.(32) over the whole region $[-E_0,E_0]$ we get an uncertainty coming from the regions $E_0-|E|\le\delta E$, where our integrand differs substantially from the exact one. To estimate an order of magnitude of this uncertainty, let us consider the contribution coming from this regions with *our* integrand. In the "worst" case of the coinciding correlator arguments $x_1=x_2$ the integrand has integrable *square root* singularities at the ends $E=\pm E_{max}$, and their contribution has a relative order of magnitude $(\delta E/E_0)^{1/2}\sim b^{-1/2}$. In the case of nonzero $x=x_2-x_1$ the contribution of of the regions $E_0-|E|\le\delta E$ is exponentially small at $x\gg b^{-1}$. In particular, we conclude that in the recent numerical BRM simulations [3] done for the matrix bandwidth $b=4-12$ substantial deviations order of $b^{-1/2}$ can occur for the correlator at $x\lesssim b^{-1}$.

The integration in eq.(32) can hardly be performed analytically. Let us therefore consider two simple limiting cases first. It can be most easily done for the correlator value at $x=(x_2-x_1)=0$:

$$\mathcal{K}_t(0)=2\mathcal{K}_E(0;E)\mid_{E=0}=\frac{2}{3l_c(0)}(1+C_\times), \qquad (35)$$

which is a "time-averaged" version of the so-called inverse participation ratio (IPR) $\xi_t\equiv\sum_\alpha|\varphi_\alpha|^4$. Note, that *in the BRM model* "time-averaged" IPR is twice as large as compared to its "fixed energy" counterpart usually considered in the solid physics and defined as $\xi_E\equiv\sum_\alpha|\varphi_\alpha|^4\,\delta(E_\alpha-E)$ for quantum states near $E=0$. It is a natural consequence of the averaging over localized quantum states with $l_c(E)<l_0(0)$.

Large-x asymptotics ($x \gg l_c(0)$) can be calculated by the steepest-descent method. Performing the calculation, we find

$$\mathcal{K}_t(x) = \left[\frac{4}{\sqrt{\pi}}\left(\frac{l_c(0)}{|x|}\right)^{1/2}\right] \mathcal{K}_E(x;E)\,|_{E=0}\,. \tag{36}$$

Comparing with the expression eq.(35) we see that at large x the extra factor reduces the value of the correlator. Therefore, converting $\mathcal{K}_E$ into $\mathcal{K}_t$ *changes* the correlator *form*, most considerably in the region $|x| \lesssim l_c(0)$.

In the recent numerical work [3] the data for the correlator $\mathcal{K}_t(x)$ have been obtained in the course of a direct numerical simulation of the BRM orthogonal ensemble. With use of the Gogolin formula [8] the best fit value for the localisation length was found $l_c \approx 0.29b^2$. In Fig.1 we plot the correlator $\mathcal{K}_t(x)$ predicted by eq.(32) which has *no free parameters*

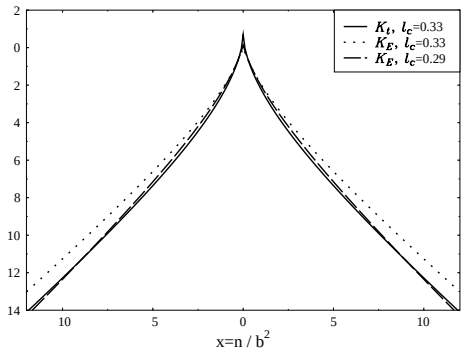

Figure 1: The *time-averaged* density-density correlator $\mathcal{K}_t(x)$ versus $x = n/b^2$. The solid line represents the prediction by eq.(32). The dashed line shows the parametrization of the numerical data [3], the Gogolin formula with $l_c = 0.29$, and dotted line corresponds the same Gogolin formula with $l_c = 0.33$.

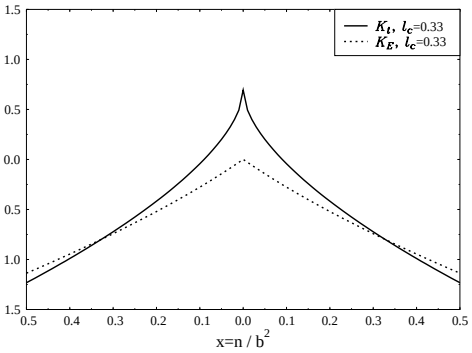

Figure 2: The *time-averaged* density-density correlator $\mathcal{K}_t(x)$ versus $x = n/b^2$ (the central region). The solid line represents prediction by eq.(32). The dotted line shows the Gogolin formula with $l_c = 0.33$.

and corresponds to $l_c(E=0) = b^2/3$ [9]. The dashed line presents the parametrization of the data by the Gogolin formula with $l_c = 0.29b^2$. To illustrate an accuracy of the agreement, we plot also the Gogolin formula with $l_c = b^2/3$. We see, that the agreement looks reasonable. For more reliable analysis one needs to compare the correlator $\mathcal{K}_t(x)$ directly with the data for BRM ensembles with the matrix bandwith $b \gg 1$.

However, the most crucial test of our theory would be the comparison of the correlator behavior at $x \lesssim 1$ with numerical data (see, Fig.2). We expect it to be essentially more peaked at $x = 0$ than the "fixed energy" correlator $\mathcal{K}_E(x_2 - x_1; E)$ given by the Gogolin formula with $l_c = b^2/3$. Apart from this fact, it would also be interesting to compare the *time-averaged* IPR with data, which is predicted to be twice as large in comparison with the corresponding "fixed energy" quantity.

6 Conclusions

In this work we have studied the density-density correlators in the BRM ensembles for the orthogonal and unitary symmetry cases. In contrast to the works [2, 5, 9, 13] we consider the case of infinite matrix size $N \to \infty$, with correlator arguments $n_1 \leq n_2$ taken well inside the argument region: $[1, ..., n_1, ..., n_2, ..., N]$, far from the ends: $n_1 \gg b^2$, $N - n_2 \gg b^2$ (b is matrix bandwidth).

We have found, that the "fixed energy" density-density correlator (2) turns out to be the same in both the orthogonal and unitary BRM ensembles, within the accuracy $O(b^{-2})$. Moreover, its functional form coincides exactly with the old Gogolin formula originally derived for 1D electron system in disordered potential.

It deserves to be mentioned that the analytical solution of the BRM model given in [5, 2] consists of several approximation steps. The first one is the mapping of the model onto the supersymmetrical σ-model. It has, probably, the worst accuracy of order $O(b^{-1/2})$. At the same time the accuracy of the σ-model solution, including the continuous approximation, is the order $O(\gamma^{-1}) \sim O(b^{-2})$. Thus, one may expect deviations from the exact solution to be of order of $O(b^{-1/2})$.

Beside the "fixed energy" density-density correlator $\mathcal{K}_E(x_2 - x_1; E)$, we have also calculated the *time-averaged* correlator $\mathcal{K}_t(x_2 - x_1)$. Two functions are found to be different, especially in the region of $(x_2 - x_1) \lesssim 1$. We predict also, that the *time-averaged* inverse participation ratio for BRM model is twice as large as the "fixed energy" one.

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[9] To facilitate the comparison with data we plot the curves in terms of scaled variable $x = n/b^2$, as defined in Ref.[3]

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