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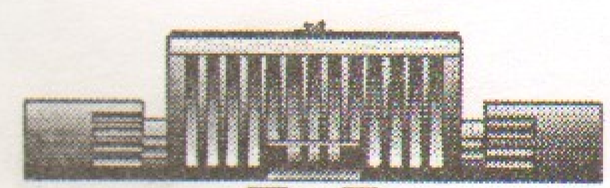

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On some properties
of the SU($N_c$) Yang-Mills vacuum state

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**Abstract**

The asymptotic behaviour of the vacuum energy density, $\overline{U}(\theta)$, at $\theta \to \pm i\infty$ is found out. A qualitative discussion and a new interpretation of the $\overline{U}(\theta)$ behaviour are presented. It is emphasized that there is no confinement at $\theta = \pi$. The potential of the monopole field condensing in the Yang-Mills vacuum is obtained.

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1. Let us consider the (Euclidean) YM $SU(N_c)$ theory defined by the integral:

$$Z = \sum_k \int dA_\mu \, \delta(Q-k) \exp\left\{-\int dx \, I_o(x)\right\}, \quad I_o = \frac{1}{4g_o^2} G_{\mu\nu}^2 , \tag{1}$$

where Q is the topological charge, and let us integrate out gluon fields with the constraint: [1]

$$Z = \int dS \, \theta(S) \exp\left\{-\int dx \, I(S)\right\}, \quad I(S) = N_c^2 S \frac{b_o}{4}\left[\ln\frac{M_o^4}{\bar\Lambda^4} + f(S, M_o^2)\right],$$

$$\exp\left\{-\int dx \, N_c^2 S \frac{b_o}{4} f(S, M_o^2)\right\} = \sum_k \int dA_\mu \, \delta(Q-k) \, \delta\left(S - \frac{G_{\mu\nu}^2}{32\pi^2 N_c}\right), \tag{2}$$

where $M_o$ is the ultraviolet cut off, and we substituted: $b_o = 11/3$, $1/g_o^2 = (N_c b_o/32\pi^2)\ln(M_o^4/\bar\Lambda^4)$.

What can be said about the dimensionless function $f(S, M_o^2)$ in eq.(2)? Let us emphasize that it does not know about $\bar\Lambda$. Thus, recalling for a renormalizability of theory, it has the form:

$$f(S, M_o^2) = -\ln\left(\frac{M_o^4}{S}\right) + C_N + D_N \frac{\partial_\mu S \, \partial_\mu S}{S^{3/2}} + \cdots \tag{3}$$

So, the partition function takes the form:

$$Z = \int dS \, \exp\left\{-N_c^2 \frac{b_o}{4}\left[S \ln\left(\frac{S}{\bar\Lambda^4 \delta_N}\right) + \right.\right.$$

$$\left.\left. + \frac{1}{S^{3/2}} \partial_\mu S \, \Delta\left(\frac{q^2 = -\partial^2}{S^{1/2}}\right) \partial_\mu S + \ldots\right]\right\}, \tag{4}$$

[1] This is what differs our approach from those proposed by G. Savvidy [1], the latter one leading to instabilities.

with $\Lambda^4 = \bar{\Lambda}^4 e^{C_o}$, the only observable dimensionful parameter of theory (and $\delta_N = \exp(C_N - C_o) \simeq (1 + O(1/N_c^2))$ at $N_c \gg 1$).

Really, the above reference to a renormalizability of theory and the explicit form of eq.(4) imply that all quantum fluctuations of the S-field are integrated out also.

Let us emphasize that, as it is seen from its derivation, the above action, eq.(4), is not some approximate effective or low energy action but the exact one, and the exact answers for all S-field correlators are obtained from this action using tree diagrams. [2]

Little is known about the inverse propagator of the S-field, except what can be obtained from the asymptotic freedom and operator expansions. Let us define the propagator: $\int dx \exp(iqx) \langle S(x) S(0) \rangle_{con} = D(q^2, \Lambda^2)$. At $q^2 \gg \Lambda^2$:

$$D(q^2, \Lambda^2) = \Big\{ \Big[ C_1\, \alpha_s(\frac{q^2}{\Lambda^2})\, q^4 \,(1 + O(\alpha_s)) \Big] + \Big[ C_2\, \alpha_s^2(\frac{q^2}{\Lambda^2})\, \Lambda^4 \,(1 + O(\alpha_s)) \Big] + \\ + \Big[ C_3\, \alpha_s^{\delta}(\frac{q^2}{\Lambda^2}) \frac{\Lambda^6}{q^2} (1 + O(\alpha_s)) \Big] + \cdots \Big\}.$$

Clearly, the behaviour of $\Delta(q^2/S^{1/2})$ in eq.(4) at $q^2 \gg S^{1/2}$ can be reconstructed from the above behaviour of D, with a replacement: $\Lambda^2 \to e^{-1/2} S^{1/2}$. [3]

In what follows we will be interested mainly in the potential part (i.e. without space-time derivatives) of the action and will ignore all terms with such derivatives.

It follows from eq.(4) that the S-field condenses in the YM-vacuum, and the values of the condensate and the vacuum energy density are:

$$\bar{S} = e^{-1} \Lambda^4 \delta_N\,, \quad \bar{U} = -N_c^2 \frac{b_o}{4} \Lambda^4 \delta_N < 0\,. \tag{5}$$

As it is, the potential in eq.(4) is well known (see e.g. [2]). However, the S-field was considered previously mainly as the effective dilaton field, i.e. the

[2] We used the one-loop $\beta$-function in the above equations. More precisely, to obtain the renormalization group invariant expressions we have to fix the renormalization group invariant quantities, e.g. $(-1/N_c^2 b_o) T_{\mu\mu} = z_G(g_o^2) S = (1 + O(g_o^2)) S$ in this case, where $T_{\mu\mu}$ is the energy-momentum tensor trace. Having this in mind, it is implied everywhere below that appropriate renormalization factors, $z_i$, are included into definitions of all fields and parameters we deal with. In eq.(2) these higher order corrections die off in the formal limit $M_o \to \infty$.

[3] The above form of the S-field propagator shows that quantum loop corrections of the S-field contribute a small amount, $\sim (1/N_c) \ln\ln M_o^2$, to the renormalization of $1/g_o^2$, i.e. a relative correction $\sim 1/N_c^2$ into the second term, $N_c b_1$, of the $\beta(g^2)$-function.



interpolating field of the lightest scalar gluonium. Besides, such Lagrangians were usually considered as some "effective" Lagrangians, the meaning of "effective" was obscure, as well as their connection with the original YM Lagrangian. Our approach allows to elucidate its real origin and meaning and, on this basis, to use it for investigation of vacuum properties. Moreover, the above described approach will be used everywhere below when dealing with other fields also.

**2.** Let us extend now our theory and add the $\theta$-term to the action $I_o$ in eq.(1), and let us integrate out gluon fields with the fields S(x) and P(x) both fixed ($\tilde{\theta} = i\theta$, $Q = N_c \int dx P$) :

$$Z = \sum_k \int dS\, \theta(S) \int dP\, \theta(S-P)\, \delta(Q-k) \exp\Big\{ -N_c^2 \Big[ I_o(S,P) - \frac{\tilde{\theta}}{N_c} P \Big] \Big\}\,,$$

$$\exp\{-N_c^2 I_o(S,P)\} = \int dA_\mu\, \delta\Big( S - \frac{G_{\mu\nu}^2}{32\pi^2 N_c} \Big)\, \delta\Big( P - \frac{G\tilde{G}}{32\pi^2 N_c} \Big)\,. \tag{6}$$

The positivity of the Euclidean integration measure (at real $\tilde{\theta}$) leads to a number of useful sign inequalities, like $\overline{S}(\tilde{\theta}) \geq 0$, $\overline{P}(\tilde{\theta}) \geq 0$, etc. Because $\overline{P}(\tilde{\theta}) = (b_o/4)\, d\overline{S}(\tilde{\theta})/d\tilde{\theta}$, this shows that $\overline{S}(\tilde{\theta})$ grows monotonically with $\tilde{\theta}$ (so that the energy density decreases monotonically with $\tilde{\theta}$).

Let us write the general form of the total action, $I = I_o - \frac{\tilde{\theta}}{N_c} P$:

$$I(S,P) = \frac{b_o}{4} S \Big[ \ln\Big( \frac{S}{\Lambda^4} \Big) + f(z = \frac{P}{S}) \Big] - \frac{\tilde{\theta}}{N_c} P\,, \tag{7}$$

and the saddle point equations:

$$\frac{\tilde{\theta}}{N_c} = \frac{b_o}{4} f'(\bar{z}), \quad \frac{b_o}{4} \Big[ \ln\Big( \frac{e\bar{S}}{\Lambda^4} \Big) + f(\bar{z}) \Big] = \frac{\tilde{\theta}}{N_c} \bar{z}. \tag{8}$$

Consider now the behaviour at $\tilde{\theta} \to \infty$. It is seen from eq.(8) that $\bar{z}(\tilde{\theta})$ approach $z_o$, - the singularity point of $f'(z)$. It looks physically unacceptable if $f(z)$ had singularities (say, poles or branch points) inside the physical region $0 < z < 1$. They can develop at the edges of the physical region only: $z \to 0$ or $z \to 1$. The behaviour of $f(z)$ at $z \to 0$ is regular however, $\sim z^2$. So, we conclude that: $\bar{z}(\tilde{\theta}) = \overline{P}(\tilde{\theta})/\overline{S}(\tilde{\theta}) \to 1$ at $\tilde{\theta} \to \infty$. Supposing that the singularity is gentle enough so that $f(1)$ is finite, one obtains then from eq.(8):

$$\bar{P}(\tilde{\theta}) \to \bar{S}(\tilde{\theta}) \to C \exp\left\{\frac{n}{b_o}\frac{\tilde{\theta}}{N_c}\right\}, \; C = e^{-f(1)-1}\Lambda^4, \;\; \tilde{\theta} \to \infty, \tag{9}$$

where $n = 4$ is the space-time dimension (compare with the $CP^N$ model, see appendix).

Based on eq.(9), a simple minded model for $I(S, P)$ in eq.(7) looks as (in Minkowsky space):

$$I(S,P) = \frac{1}{2}\left\{(S+iP)\ln\left(\frac{S+iP}{\Lambda^4 e^{i\theta/N_c}}\right) + h.c.\right\} - \frac{1}{12} S \ln\left(\frac{S}{\Lambda^4}\right). \tag{10}$$

When integrated over the S and P fields it gives (at the space-time volume $V \to \infty$):

$$\bar{S}(\theta) = e^{-1}\Lambda^4 \left[\cos^{4/b_o}\left(\frac{\theta}{N_c}\right)\right]_{2\pi}, \; \bar{U} = -N_c^2 \frac{b_o}{4}\bar{S}(\theta), \; \bar{P}(\theta) = -\frac{1}{N_c}\frac{d\bar{U}(\theta)}{d\theta}. \tag{11}$$

Here, the notation $[f(\theta/N_c)]_{2\pi}$ means that this function is $f(\theta/N_c)$ at $-\pi \le \theta \le \pi$, and is glued then to be periodic in $\theta \to \theta + 2\pi k$, i.e.:

$$\left[f(\frac{\theta}{N_c})\right]_{2\pi} = \min_k f\left(\frac{\theta + 2\pi k}{N_c}\right). \tag{12}$$

Evidently, the above periodically glued structure of the vacuum energy density results from the quantization of the topological charge in our theory, and is not connected with a concrete form of the action $I_o(S, P)$ in eq.(6).

The above model does not pretend to be the exact answer, but it is simple, has a reasonable qualitative behaviour in the whole complex plane of $\theta$, and obeys the right asymptotic behaviour at $\theta \to \pm i\infty$.

It was inspired by a "different status" of two parts in $b_o/4 = (1 - \frac{1}{12})$. The first part is connected with zero mode contributions in the instanton background, and the analytic in $\chi = (S + iP)$ term in eq.(10) is expected to be connected with this part of $b_o$ only, while the second, nonanalytic in $\chi$, part of eq.(10) is expected to be due to the "nonzero mode part" of $b_o$ only.

In Euclidean space, one has from eq.(11):

$$\bar{S}(\tilde{\theta}) = e^{-1}\Lambda^4 \cosh^{4/b_o}\left(\frac{\tilde{\theta}}{N_c}\right), \quad \bar{P}(\tilde{\theta}) = \tanh\left(\frac{\tilde{\theta}}{N_c}\right)\bar{S}(\tilde{\theta}), \tag{13}$$



so that the asymptotic behaviour, eq.(9), is reproduced, and $\bar{z}(\theta) = \bar{P}/\bar{S} \to 1$ at large $\tilde{\theta}$. In principle, the equation (13) can be checked in lattice calculations. [4]

One point is worth mentioning in connection with the asymptotic behaviour, eq.(9). Based on the quasiclassical (one loop) instanton calculations [3], one would expect the following qualitative picture. In $SU(N_c)$, each instanton splits up into $N_c$ "instantonic quarks" which appear as appropriate degrees of freedom in the dense instanton ensemble. As a result, each instantonic quark carries the factor $\exp\{i\theta/N_c\}$ in its density. So, one would expect the behaviour $\bar{P}(\tilde{\theta}) \to \bar{S}(\tilde{\theta}) \sim \exp\{\tilde{\theta}/N_c\}$ at large $\tilde{\theta}$, in disagreement with eq.(9). We conclude that something is missing here in the above picture of instantonic quarks, even in its $\theta$-dependence.

**3.** We will describe now a new qualitative interpretation of the (glued) periodicity property of vacuum energy density, $\bar{U}(\theta)$. Let us consider $N_c = 2$ for simplicity, and let us suppose the "standard picture" of the confinement mechanism to be valid. I.e., the internal (dynamical) Higgs breaking $SU(2) \to U(1)$ takes place and, besides, the $U(1)$ magnetic monopoles condense. As shown by E.Witten [4], the monopoles turn into the dyons with the $U(1)$ electric charge $\theta/2\pi$ when the $\theta$ - term is introduced. Although, strictly speaking, the Witten result is applicable in the quasiclassical region only, because the effect is of a qualitative nature, there are all the reasons to expect it will survive in the strong coupling region also ( and only the units of the electric and magnetic charges will change).

So, there are pure monopoles and antimonopoles with the magnetic and electric charges $(g, e) = (1, 0)$ and $(\bar{g}, \bar{e}) = (-1, 0)$ in the condensate at $\theta = 0$, while they turn into the dyons and antidyons with the charges $d_1^\theta = (1, \theta/2\pi)$ and $\bar{d}_1^\theta = (-1, -\theta/2\pi)$ respectively, as $\theta$ starts to deviate from zero. As a result, the vacuum energy density begins to encrease, as it follows from general considerations.

It is a special property of our system that there are two types of condensates made of the dyons and antidyons with the charges: $\{(1, 1/2); (-1, -1/2)\}$ and $\{(1, -1/2); (-1, 1/2)\}$, and having the same energy density. Thus, the vacuum becomes twice degenerate at $\theta \to \pi$, so that the branch changing can take place if this will lower the energy density. [5] And indeed it lowers, and

[4] There is another simple model: $(1/N_c^2)U = \frac{1}{2}\left\{\phi \ln \frac{\phi}{\Lambda^4 e^{i\theta/N_c}} + h.c.\right\}$, $\phi = (b_o/4)S + iP$, leading to $\bar{U}(\theta) \sim [\cos(\frac{b_o}{4}\frac{\theta}{N_c})]_{2\pi}$, i.e. with the same asymptotic behaviour at $\theta \to \pm i\infty$. The problem with it is that there are reasons to expect that the vacuum energy is exactly zero at $\theta = \pi$ and $N_c = 2$, and this model does not fulfil this, while eq.(11) does. Besides, we see no reasons here for the potential to be analytic function of one variable.

[5] Let us emphasize that the existence of two degenerate vacuum states at $\theta = \pi$ does not

this leads to a casp in $\bar{U}(\theta)$. At $\theta > \pi$, the vacuum is filled now with the new dyons with the charges: $d_2^\theta = (1, -1 + \theta/2\pi)$, $\bar{d}_2^\theta = (-1, 1 - \theta/2\pi)$. As $\theta$ increases further, the electric charge of the $d_2^\theta$ -dyons decreases, and the vacuum energy density decreases with it. Finally, at $\theta = 2\pi$, the $d_2^\theta$-dyons (which were the (1,-1)-dyons at $\theta = 0$) become the pure monopoles, and the vacuum state becomes exactly as it was at $\theta = 0$, i.e. the same condensate of pure monopoles and antimonopoles.

We empasize that, as it follows from the above picture, it is wrong to imagine the vacuum state at $\theta = 2\pi$ as, for instance, the condensate of the dyons with the charges (1,-1), degenerate in energy with the pure monopole condensate at $\theta = 0$. [6]

In fact, for the above described mechanism to be operative, there is no need to trace the real dynamical picture underlying the level crossing occuring in the infinitesimal vicinity of $\theta = \pi$. Formally, it is sufficient to say: "there is a possibility for the charged electrical degrees of freedom to rearrange themselves without changing the energy". However, it may be useful to have a more visible picture of what really happens, even if it is a pure speculation. So, we can imagine that when $\theta \to \pi$ and the vacuum becomes unstable, the charged gluons are copiously produced. The gluon with the charges (0,-1) and the $d_1 = (1, 1/2)$ - dyon form the "bound state" which is the $d_2 = (1, -1/2)$ - dyon. [7]

Some analogy with the simplest Schwinger model may be useful at this

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follow from symmetry considerations alone (like $\bar{U}(\theta) = \bar{U}(-\theta)$ and $\bar{U}(\theta) = \bar{U}(\theta + 2\pi k)$). It is sufficient to give a counterexample. So, let us consider the Georgy-Glashow model, with the large Higgs vacuum condensate resulting in $SU(2) \to U(1)$. In this case, the $\theta$-dependence of the vacuum energy density is due to a rare quasiclassical instanton gas, and is $\sim \cos(\theta)$. All the above symmetry properties are fulfilled, but there is only one vacuum state at $\theta = \pi$.

[6] In this respect, the widely used terminology naming two singularity points, $u = \pm\Lambda^2$, on the $\mathcal{N} = 2$ $SU(N_c = 2)$ SYM moduli space as those where the monopoles and, respectively, the dyons become massless, is not adequate. Indeed, let us start from the vacuum $u = \Lambda^2$ where, by definition, the massless particles are pure monopoles, and let us move along a circle to the point $u = -\Lambda^2$. On the way, the former massless monopole increases its mass because it becomes the $d_1^\theta = (1, \theta/2\pi)$ - dyon, while the former massive $d_2^o = (1, -1)$ - dyon diminishes its mass as it becomes the $d_2^\theta = (1, -1 + \theta/2\pi)$ - dyon. When we reach the point $u = -\Lambda^2$, i.e. $\theta = 2\pi$, the former dyon becomes massless, just because it becomes the pure monopole here. So, an observer living in this world will see massless monopoles, not dyons.

[7] It seems that a qualitative picture of the dyon as the bound state of the monopole and gluon is not meaningless even in the quasiclassical region. Here, although the dyon mass is: $M_d = M_{mon}(1 + O(\alpha^2))$, while the ratio of the gluon and monopole masses is $O(\alpha)$, the interaction of the gluon magnetic moment with the monopole magnetic field is capable to cancel the contribution $O(\alpha M_{mon})$ to the dyon mass.



point. Let us consider first the pure $QED_2$, without finite mass charged particles, and let us put two infinitely heavy "quarks" with the charges $\pm\theta/2\pi$ (in units of some $e_o$) at the edges of our (infinite length) space. It is well known that this is equivalent to introducing the $\theta$-angle into the $QED_2$ Lagrangian. As a result, there is the empty vacuum at $\theta = 0$, and the long range Coulomb "string" at $\theta \neq 0$. The vacuum energy density increases as: $E(\theta) = C_o\,\theta^2$, $C_o = const$, at any $0 \leq \theta < \infty$.

Let us add now some finite mass, $m \gg e_o$, and of unit charge $e_o$ field $\psi$ to the Lagrangian. When there are no external charges, this massive charged field can be integrated out, resulting in a small charge renormalization. But when the above "quarks" are introduced, the behaviour of $E(\theta)$ becomes nontrivial: $E(\theta) = C_o \min_k(\theta + 2\pi k)^2$. So, $E(\theta) = C_o\,\theta^2$ at $0 \leq \theta \leq \pi$, and $E(\theta) = C_o\,(2\pi - \theta)^2)$ at $\pi \leq \theta \leq 2\pi$.

The reason is clear. The external "quark" charge becomes "1/2" at $\theta = \pi$. At this point, the pair of $\psi$- particles is produced from the vacuum, and they separate so that to recharge the external "quarks": $\pm 1/2 \to \mp 1/2$. As a result of this rechargement, there appears a cusp in $E(\theta)$, and $E(\theta)$ begins to decrease at $\theta > \pi$, so that the former "empty" vacuum is reached at $\theta = 2\pi$.

Let us return however to our dyons. Clearly, at $0 \leq \theta < \pi$, the condensate made of only the $d_1^\theta = (1, \theta/2\pi)$ - dyons (recalling also for a possible charged gluon pair production) can screen the same type $d_k^\theta = (1, k + \theta/2\pi)$ - test dyon only ($k = 0, \pm 1, \pm 2, \ldots$; and the same for the $d_2^\theta$ - dyons at $\pi < \theta \leq 2\pi$). So, the heavy quark-antiquark pair will be confined at $\theta \neq \pi$.

Surprisingly, there is no confinement at $\theta = \pi$. The reason is that there is now an equal mixture of four species of dyons with the charges: $d_1 = (1, 1/2)$, $\bar{d}_1 = (-1, -1/2)$, $d_2 = (1, -1/2)$, and $\bar{d}_2 = (-1, 1/2)$, and this mixture is able to screen any charge, polarizing itself appropriately. To have a visible picture, we can consider the pair of $d_1 = (1, 1/2)$- and $\bar{d}_2 = (-1, 1/2)$ - dyons as being combined into one purely electrically charged "particle", so that the condensate can be viewed as consisting of these $e = (0, 1)$ and $\bar{e} = (0, -1)$ "particles", and it can screen easily the electric charge of the heavy test quark (and to diminish in this way the energy of the state, because a string is not formed). Another variant is that the $d_2$ - dyon (or $\bar{d}_1$ - antidyon) will combine with the quark to form an electrically neutral object. But this object obeys then the magnetic charge. The condensate can now be viewed as consisting of the pure "monopoles" composed from the $(d_1 d_2)$ pairs (and "antimonopoles" respectively), and it will screen the magnetic field of the above object.

However, if in the vacuum with some $\theta \neq \pi$ the finite size ball surrounding a quark and consisting of any mixture of dyons and antidyons of any possible kind is excited, it will be unable to screen the quark charge as there is no border at infinity where the residual polarization charge will be pushed out.

We have to make a reservation about the above described picture. As was pointed out above (see footnote 4), there are reasons to expect that the point $\theta = \pi$ is very especial just for $SU(N_c = 2)$, because the vacuum energy density is likely to be exactly zero here. In this case, it is natural if the dyon condensate also approaches zero therein. But theory is expected then to have massless charged particles, etc. Clearly, there is no confinement in this case also.

As for $SU(N_c > 2)$, the above described picture goes through without changes, if we suppose that $SU(2) \to U(1)$ is replaced by $SU(N_c) \to U(1)^{N_c - 1}$, and there are $(N_c - 1)$ types of monopoles in the condensate, with the magnetic charges $(m^A)_i = \pm n_i^A$, $A = 1, ..., N_c - 1$, $i = 1, ..., N_c$. (For instance, it has been proposed by 't Hooft [5] that good candidates are the "minimal monopoles" having two consecutive and opposite charges, i.e. $n_i^A = \delta_i^A - \delta_{i+1}^A$ in this case). Now, when the $\theta$- term is introduced into the Lagrangian, each one of these $(N_c - 1)$ monopoles will turn into the dyon with the same type electric charge: $(e^A)_i = \pm(\theta/2\pi)\, n_i^A$. So, the above described picture is applicable without changes. As $\theta \to \pi$, the gluons with the electric charges $(g^A)_i = \pm n_i^A$ are copiously produced to recharge these dyons, and there is no confinement at $\theta = \pi$.

4. Let us point out now that the assumption about the confinement property (at $\theta \neq \pi$) of the $SU(N_c)$ YM theory is not a pure guess, as the above discussed nonanalytical (i.e. glued) structure of the vacuum energy density, $\bar{U}(\theta)$, is a clear signal about a phase transition at some finite temperature. Indeed, at high temperatures the $\theta$-dependence of the free energy density is under control and is: $\sim T^4 (\Lambda/T)^{N_c b_o} \cos(\theta)$, due to a rare instanton gas. It is important for us here that it is perfectly analytic in $\theta$, and that this $\sim \cos(\theta)$-behaviour is T-independent, i.e. it persists with decreasing temperature. On the opposite side, at $T = 0$, the $\theta$- dependence is nonanalytic and, clearly, this nonanalyticity survives at small temperatures. So, there should be a phase transition (confinement - deconfinement) at some critical temperature, $T_c$, where the $\theta$- dependence changes qualitatively.

Finally, let us show that, supposing that the monopoles indeed condense, we can find out the form of the monopole field potential. So ($N_c = 2$ and $\theta = 0$ for simplicity), let us return to the original partition function, eq.(1), and let us suppose that we have integrated out gluons with two fields fixed: this time the field S and the monopole fields $M$ and $\overline{M}$. The potential will



have the form:

$$U(S, M, \overline{M}) = b_o S \ln\left(\frac{S}{\Lambda^4}\right) - S f\left(z = \frac{\overline{M} M}{S^{1/2}}\right), \tag{14}$$

and the saddle point equation is: $f'(z) = 0$. By the above assumption, the function $f(z)$ is such that this equation has a nontrivial solution: $(\overline{M} M) = z_o S^{1/2}$, i.e. with $z_o \neq 0$. Substituting it now back to eq.(14), we obtain the monopole field potential:

$$U(M, \overline{M}) = \frac{2b_o}{z_o^2} \left(\overline{M} M\right)^2 \ln\left(\frac{\overline{M} M}{\Lambda_M^2}\right), \quad \Lambda_M^2 = \Lambda^2 z_o e^{-f(z_o)/2b_o}. \tag{15}$$

We see that the assumption made is selfconsistent, i.e. the monopole field indeed condenses. It is sufficient to supply this potential with the simplest kinetic terms of the monopole and the (dual) neutral gluon fields, to obtain the explicit solution for the electric string.


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### Appendix

The purpose of this appendix is to present in an explicit form the asymptotic behaviour, at $\theta \to \pm i\infty$, of the vacuum energy density of the $CP^N$-model (the leading term at $N \gg 1$).

The partition function can be written (in Euclidean space) in the form:

$$Z = \int dn d\bar{n} dA_\mu d\lambda \exp\left(-\int dx I(x)\right),$$

$$I = \left\{\overline{D_\mu n}\, D_\mu n - U \bar{n} n + \frac{N}{f_o} U - \frac{\tilde{\theta}}{2\pi} F\right\},$$

where:

$$D_\mu = \partial_\mu + iA_\mu,\; Q = \frac{F}{2\pi} = \frac{1}{2\pi}\epsilon_{\mu\nu}\partial_\mu A_\nu,\; \frac{-1}{N} T_{\mu\mu} = \frac{1}{2\pi} U = \frac{f_o}{2\pi N} \overline{D_\mu n}\, D_\mu n,$$

$U = -i\lambda$, $\tilde{\theta} = i\theta$, and $f_o$ is the bare coupling: $f_o^{-1} = (b_o/4\pi)\ln(M_o^2/\Lambda^2)$, $b_o = 1$. Integrating out the n-field, one obtains the action:

$$\frac{1}{N}I = Tr\ln\left(-D_\mu^2 - U\right) + \frac{U}{4\pi}\ln\left(\frac{M_o^2}{\Lambda^2}\right) - \frac{\tilde{\theta}}{N_c}\frac{1}{2\pi}F.$$

As we need the potential only, the fields $U$ and $F$, which are direct analogs of the S and P fields in the YM theory, can be considered as constant ones. The determinant for this case was calculated by F.Riva [6]:

$$Tr\ln\left(-D_\mu^2 - U\right) = \frac{-1}{4\pi}\int_{1/M_o^2}^{\infty}\frac{dt}{t}\left[\frac{F}{\sinh(Ft)}\exp\{Ut\} - \frac{1}{t}\right].$$

The saddle point equations are:

$$\ln\left(\frac{M_o^2}{\Lambda^2}\right) = \int_{1/M_o^2}^{\infty} dt\,\exp\{\overline{U}t\}\,\frac{\overline{F}}{\sinh(\overline{F}t)}, \qquad (a1)$$

$$\int_0^{\infty}\frac{dz}{z\sinh(z)}\exp\{\frac{\overline{U}}{\overline{F}}z\}\left[z\coth(z) - 1\right] = 2\frac{\tilde{\theta}}{N_c}. \qquad (a2)$$

We obtain from (a2) at $\tilde{\theta} \to \infty$:

$$\frac{\overline{U}(\tilde{\theta})}{\overline{F}(\tilde{\theta})} = 1 - \Delta(\tilde{\theta}), \quad \Delta(\tilde{\theta}) = \frac{N}{\tilde{\theta}}\left[1 - \frac{N}{\tilde{\theta}}\ln\frac{\tilde{\theta}}{N} + O(\frac{N}{\tilde{\theta}})\right],$$

and from (a1):

$$\ln\left(\frac{M_o^2}{\Lambda^2}\right) = \ln\left(\frac{M_o^2}{\overline{F}(\tilde{\theta})}\right) + \frac{2}{\Delta(\tilde{\theta})} + O(1); \quad \overline{F}(\tilde{\theta}) \to const\left[\frac{\tilde{\theta}}{N_c}\exp\left\{\frac{\tilde{\theta}}{N_c}\right\}\right]^{d/b_o},$$

where d=2 is the space-time dimension, and $b_o = 1$.

So, the behaviour of the vacuum energy density, $(1/2)T_{\mu\mu} = -N\overline{U}(\tilde{\theta})/4\pi$, is nontrivial. $\overline{U}(\tilde{\theta})$ starts with the negative value $\sim(-\Lambda^2)$ at $\tilde{\theta} = 0$, then increases monotonically with increasing $\tilde{\theta}$, passes zero at some $\tilde{\theta}_o$, and becomes positive and large, approaching $\overline{F}(\tilde{\theta})$ from below at large $\tilde{\theta}$.



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