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ON THE NATURE  
OF COULOMB CORRECTIONS  
TO THE  $e^+e^-$  PAIR PRODUCTION  
IN ULTRARELATIVISTIC  
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**On the nature of Coulomb corrections to the  $e^+e^-$  pair  
production in ultrarelativistic heavy-ion collisions**

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**Abstract**

We manifest the origin of the wrong conclusion made by several groups of authors on the absence of Coulomb corrections to the cross section of the  $e^+e^-$  pair production in ultrarelativistic heavy-ion collisions. The source of the mistake is connected with an incorrect passage to the limit in the expression for the cross section. When this error is eliminated, the Coulomb corrections do not vanish and agree with the results obtained within the Weizsäcker-Williams approximation.

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RHIC and LHC projects initiated a set of recent publications on the  $e^+e^-$  pair production in ultrarelativistic heavy-ion collisions. Using slightly different approaches, the authors of [1, 2, 3] calculated the cross section of the process exactly in the parameters  $\alpha Z_{A,B}$  ( $Z_{A,B}$  being the charge numbers of the nuclei  $A$  and  $B$ ,  $\alpha$  is the fine-structure constant). In these papers the nuclei were treated as sources of the external field, and the amplitude was calculated at a fixed impact parameter of the nuclei. After that the cross section was obtained by the integration over the impact parameter. As a result, the conclusion was made that the exact cross section coincides with that calculated in the lowest order perturbation theory with respect to  $\alpha Z_{A,B}$  (Born cross section). On the other hand, in the Weizsäcker-Williams approximation with respect to one of the nuclei, the cross section of the process is proportional to the well-known cross section of the  $e^+e^-$  pair production by a photon in a Coulomb field [4] and, therefore, contains the Coulomb corrections. This obvious circumstance was observed in [5], where the Coulomb corrections in the process under discussion were calculated. Though the existence of the Coulomb corrections is out of doubt, the source of the disagreement between the results was not revealed so far. This question is important from the theoretical point of view, since the approach developed in [1, 2, 3] is used now in QCD. In the present paper we present the solution of this puzzle.

Let the ultrarelativistic nuclei  $A$  and  $B$  move in the positive and negative directions of the  $z$  axis, respectively. Then the expression for the cross section

of the  $e^+e^-$  pair production, obtained in [1, 2, 3], reads

$$d\sigma = \frac{m^2 d^3 p d^3 q}{(2\pi)^6 \varepsilon_p \varepsilon_q} \int \frac{d^2 k}{(2\pi)^2} |F_B(\mathbf{k})|^2 |F_A(\mathbf{q}_\perp + \mathbf{p}_\perp - \mathbf{k})|^2 |\mathcal{M}(\mathbf{k})|^2, \quad (1)$$

$$\mathcal{M}(\mathbf{k}) = \bar{u}(p) \frac{\boldsymbol{\alpha}(\mathbf{k} - \mathbf{p}_\perp) + \gamma_0 m}{-p_+ q_- - (\mathbf{k} - \mathbf{p}_\perp)^2 - m^2 + i\epsilon} \gamma_- u(-q) +$$

$$+ \bar{u}(p) \frac{-\boldsymbol{\alpha}(\mathbf{k} - \mathbf{q}_\perp) + \gamma_0 m}{-p_- q_+ - (\mathbf{k} - \mathbf{q}_\perp)^2 - m^2 + i\epsilon} \gamma_+ u(-q).$$

Here  $\mathbf{p}$  and  $\varepsilon_p$  ( $\mathbf{q}$  and  $\varepsilon_q$ ) are the momentum and energy of the electron (positron),  $u(p)$  and  $u(-q)$  are positive- and negative-energy Dirac spinors,  $\boldsymbol{\alpha} = \gamma^0 \boldsymbol{\gamma}$ ,  $\gamma_\pm = \gamma^0 \pm \gamma^z$ ,  $\gamma^\mu$  are the Dirac matrices,  $p_\pm = \varepsilon_p \pm p^z$ ,  $q_\pm = \varepsilon_q \pm q^z$ ,  $m$  is the electron mass,  $\mathbf{k}$  is a two-dimensional vector lying in the  $xy$  plane, and the function  $F(\boldsymbol{\Delta})$  is proportional to the electron eikonal scattering amplitude in the potential  $V(\mathbf{r})$  of the corresponding nucleus:

$$F(\boldsymbol{\Delta}) = \int d^2 \rho \exp[-i\boldsymbol{\rho}\boldsymbol{\Delta}] \{ \exp[-i\chi(\boldsymbol{\rho})] - 1 \}, \quad \chi(\boldsymbol{\rho}) = \int_{-\infty}^{\infty} dz V(z, \boldsymbol{\rho}). \quad (2)$$

For the potential  $V(\mathbf{r}) = V_c(r) = -Z\alpha/r$ , the integral in  $\chi(\boldsymbol{\rho})$  becomes divergent and requires a regularization. This regularization can be made by using the potential  $V(\mathbf{r}) = -Z\alpha \exp(-r/a)/r$ . Performing the integration in (2), and taking the limit  $a \rightarrow \infty$  at fixed  $\boldsymbol{\Delta} \neq 0$ , one obtains (up to the constant phase depending on  $a$ ):

$$F(\boldsymbol{\Delta}) = \mathcal{F}(\boldsymbol{\Delta}) \equiv i\pi Z\alpha \frac{\Gamma(1 - iZ\alpha)}{\Gamma(1 + iZ\alpha)} \left( \frac{4}{\Delta^2} \right)^{1 - iZ\alpha}. \quad (3)$$

Actually, to obtain this result one can use any regularization of the phase  $\chi(\boldsymbol{\rho})$  for which  $\chi(\boldsymbol{\rho}) \rightarrow 0$  at  $\rho \rightarrow \infty$ . Since  $|\mathcal{F}(\boldsymbol{\Delta})|^2 = (4\pi Z\alpha/\Delta^2)^2 \propto Z^2$ , then the substitution (3) into (1) would lead to the wrong conclusion [1, 2, 3] that the exact cross section coincides with the Born result. Let us show that, in order to obtain the Coulomb corrections in (1), it is necessary first to take the integral over  $\mathbf{k}$  using the functions  $F(\boldsymbol{\Delta})$  with the regularized phase and then remove the regularization.

Consider the integral

$$G = \int \frac{d^2k}{(2\pi)^2} k^2 (|F(\mathbf{k})|^2 - |F^0(\mathbf{k})|^2) , \quad (4)$$

where  $F^0(\mathbf{\Delta}) = -i \int d\boldsymbol{\rho} \exp(-i\mathbf{\Delta}\boldsymbol{\rho})\chi(\boldsymbol{\rho})$  is the first term of the expansion of  $F(\mathbf{\Delta})$  with respect to the potential. For  $F = \mathcal{F}$  and, correspondingly,  $F^0 = \mathcal{F}^0 \equiv 4i\pi Z\alpha/\Delta^2$ , the integrand in (4) vanishes. Let us show that the integral  $G$  is not equal to zero for the regularized  $F$  and is independent of the regularization method, if  $V(\mathbf{r}) \rightarrow -Z\alpha/r$  at  $r \rightarrow 0$  ( when  $\chi(\boldsymbol{\rho}) \rightarrow 2Z\alpha \ln(\rho) + const$  at  $\boldsymbol{\rho} \rightarrow 0$ ). For the sake of simplicity, we present the proof of this statement for a spherically symmetric potential  $V(r)$ . Taking the integral in (2) over the angle of  $\boldsymbol{\rho}$ , and integrating by parts over  $\rho$ , we obtain the following expression for  $F$ :

$$F(\mathbf{k}) = \frac{2\pi i}{k} \int_0^\infty d\rho \rho J_1(k\rho)\chi'(\rho) \exp(-i\chi(\rho)) , \quad (5)$$

where  $J_1(x)$  is the Bessel function. The function  $F^0(\mathbf{k})$  can be obtained from (5) by omitting the exponent in the integrand. Substituting (5) into (4), and integrating over the angle of  $\mathbf{k}$ , we find

$$G = 2\pi \int_0^\infty dk k \int_0^\infty \int_0^\infty d\rho_1 d\rho_2 \rho_1 \rho_2 J_1(k\rho_1) J_1(k\rho_2) \times \\ \times \chi'(\rho_1)\chi'(\rho_2) \{ \exp[-i\chi(\rho_1) + i\chi(\rho_2)] - 1 \} . \quad (6)$$

If one changes naively the order of integration over  $k$  and  $\rho_{1,2}$  and takes the integral over  $k$ , using the relation

$$\int_0^\infty dk k J_1(k\rho_1) J_1(k\rho_2) = \frac{1}{\sqrt{\rho_1\rho_2}} \delta(\rho_1 - \rho_2) ,$$

then, after the integration over  $\rho_1$ , the result will be zero. To demonstrate that the change of the integration order in (6) is invalid, we restrict the

upper limit of the integral over  $k$  by  $Q$ . After that one can change the order of integration in triple integral in (6). Integrating over  $k$ , we obtain

$$G = 2\pi \int_0^\infty \int_0^\infty d\rho_1 d\rho_2 \frac{Q\rho_1\rho_2}{\rho_1^2 - \rho_2^2} [\rho_2 J_0(Q\rho_2)J_1(Q\rho_1) - \rho_1 J_0(Q\rho_1)J_1(Q\rho_2)] \times \\ \times \chi'(\rho_1)\chi'(\rho_2) \{\exp[-i\chi(\rho_1) + i\chi(\rho_2)] - 1\} . \quad (7)$$

Substituting  $\rho_{1,2} \rightarrow \rho_{1,2}/Q$ , and taking the limit  $Q \rightarrow \infty$  with the use of the asymptotics of  $\chi$ , we find

$$G = 8\pi(Z\alpha)^2 \int_0^\infty \int_0^\infty d\rho_1 d\rho_2 \frac{1}{\rho_1^2 - \rho_2^2} [\rho_2 J_0(\rho_2)J_1(\rho_1) - \rho_1 J_0(\rho_1)J_1(\rho_2)] \times \\ \times \left\{ \left( \frac{\rho_2}{\rho_1} \right)^{2iZ\alpha} - 1 \right\} . \quad (8)$$

Making the change of variables  $\rho_{1,2} = r \exp(\pm t/4)$ , and integrating over  $r$ , we obtain the non-zero result for the quantity  $G$ :

$$G = 8\pi(Z\alpha)^2 \int_0^\infty dt \frac{\cos(Z\alpha t) - 1}{\exp(t) - 1} \quad (9) \\ = -8\pi(Z\alpha)^2 [\operatorname{Re}\psi(1 + iZ\alpha) + C] = -8\pi(Z\alpha)^2 f(Z\alpha) ,$$

where  $C$  is the Euler constant,  $\psi(x) = d \ln \Gamma(x) / dx$ . Thus, we come to the remarkable statement: although the main contribution to the integral in (4) comes from the region of small  $k$ , where  $|F(\mathbf{k})|$  differs from  $|\mathcal{F}(\mathbf{k})| = 4\pi Z\alpha/k^2$  and depends on the regularization parameters (the radius of screening), nevertheless, the integral  $G$  itself is a universal function of  $Z\alpha$ . Note that the integral (4) appears in the theory of multiple scattering (see [6] where the approximate formula for this integral was obtained).

Now it is clear, how to derive the Coulomb corrections starting from the expression (1). Let us calculate the Coulomb corrections related to the nucleus  $B$  (the contribution of the higher order perturbation theory with respect to the parameter  $Z_B\alpha$ ). For this purpose one should replace in (1) the

functions  $|F_B|^2$  and  $|F_A|^2$  with  $|F_B|^2 - |F_B^0|^2$  and  $|\mathcal{F}_A^0|^2$ , respectively, keeping the regularization in the functions  $F_B$  and  $F_B^0$ . The main contribution to the integral is given by the region of small  $\mathbf{k}$ . Therefore, we can neglect  $\mathbf{k}$  in the argument of  $\mathcal{F}_A^0$  and expand the matrix element  $\mathcal{M}$  at small  $\mathbf{k}$ :

$$\mathcal{M}(\mathbf{k}) \approx \mathbf{kL} , \quad (10)$$

$$\mathbf{L} = \bar{u}(p) \left\{ \frac{\boldsymbol{\alpha} (\gamma_-/p_+ - \gamma_+/q_+)}{(p_- + q_-)} + \frac{2\gamma_- (\mathbf{p}_\perp/p_+ - \mathbf{q}_\perp/q_+)}{(p_- + q_-)^2} \right\} u(-q) .$$

Using (9) and (10), and performing the summation over electron and positron polarizations, we obtain the following expression for the Coulomb corrections related to the nucleus  $B$ :

$$d\sigma_B^c = \frac{2G_B d^3p d^3q}{(2\pi)^6 \varepsilon_p \varepsilon_q} \frac{|\mathcal{F}_A^0(\mathbf{p}_\perp + \mathbf{q}_\perp)|^2}{[p_+ q_+ (p_- + q_-)]^2} \times \left\{ p_+ q_+ (\mathbf{p}_\perp + \mathbf{q}_\perp)^2 - \frac{2(\mathbf{p}_\perp q_+ q_- + \mathbf{q}_\perp p_+ p_-)^2}{(p_- + q_-)^2} \right\} . \quad (11)$$

Here  $G_B$  denotes the function  $G$  in (9) at  $Z = Z_B$ . The Coulomb corrections related to the nucleus  $A$  can be obtained from (11) by the substitution  $Z_A \leftrightarrow Z_B$  and the replacement of indices  $- \leftrightarrow +$ .

It is necessary to note the following circumstance. Actually, in the expansion over  $Z_A \alpha$  and  $Z_B \alpha$  of the differential cross section  $d\sigma/d\mathbf{p}d\mathbf{q}$  in (1), only the lowest (Born) term is correct. As for the higher order terms in (1) (Coulomb corrections), they give the correct result only after the integration over the directions of the positron (electron) momentum. This is due to the fact that the asymptotic form of the wave functions in [1, 2, 3] corresponds to the problem of scattering, but not to the problem of pair production. If one calculates the cross section integrated over the direction of  $\mathbf{q}$ , then, due to the completeness relation, it is possible to replace the set of functions containing in asymptotics the converging spherical wave with the set of functions containing the diverging spherical wave. Thus, (11) should be integrated over the angles of  $\mathbf{q}(\mathbf{p})$ . The same trick was made at the recalculation of the

bremsstrahlung cross section integrated over the photon momentum from the cross section of pair photoproduction integrated over the positron momentum [7]. It explains why the Coulomb corrections (11) are given by the region of small  $\mathbf{k}$ , while at the calculation of the Coulomb correction using the wave functions with the correct asymptotic behavior the main contribution would come from the region  $k \sim m$ . The same situation occurs at the calculation of bremsstrahlung and pair photoproduction cross sections, where the Coulomb corrections come from different regions of momentum transfers.

Let us calculate within the logarithmic accuracy the Coulomb corrections to the cross section  $d\sigma/d\varepsilon_p d\varepsilon_q$  at  $\varepsilon_{p,q} \gg m$ . At the integration over the transverse momenta the main contribution comes from the region  $\Delta = |\mathbf{p}_\perp + \mathbf{q}_\perp| \ll p_\perp, q_\perp \sim m$ . The integral over  $\Delta$  requires regularization at  $\Delta \rightarrow 0$ . It is obvious that the lower limit of integration over  $\Delta$  coincides with that in the Weizsäcker-Williams method. In the rest frame of the nucleus  $B$  it has the form  $\Delta_{min} = (\varepsilon_p^0 + \varepsilon_q^0)/\tilde{\gamma}$ , where  $\varepsilon_{p,q}^0$  are the energies of the electron and positron,  $\tilde{\gamma}$  is the Lorentz factor of the nucleus  $A$  in this frame. In the laboratory frame, where the nuclei  $A$  and  $B$  have the Lorentz factors  $\gamma_A$  and  $\gamma_B$ , respectively, one has  $\Delta_{min} = (p_+ + q_+)/\gamma_A$ . Using this cutoff, we obtain

$$d\sigma_B^c = -\frac{4}{\pi m^2} (Z_A \alpha)^2 (Z_B \alpha)^2 f(Z_B \alpha) \frac{d\varepsilon_p d\varepsilon_q}{(\varepsilon_p + \varepsilon_q)^2} \left( 1 - \frac{4\varepsilon_p \varepsilon_q}{3(\varepsilon_p + \varepsilon_q)^2} \right) \times \quad (12)$$

$$\times \left[ \ln \frac{m^2}{\Delta_{1min}^2} + \ln \frac{m^2}{\Delta_{2min}^2} \right].$$

The sum of logarithms in this formula corresponds to the contributions of two kinematic regions:  $p^z, q^z > 0$  and  $p^z, q^z < 0$ . In the first case  $\Delta_{1min} = (\varepsilon_p + \varepsilon_q)/\gamma_A$ , and the corresponding term in (12) is valid at  $m \ll \varepsilon_{p,q} \ll m\gamma_A$ . In the second case  $\Delta_{2min} = m^2/(\varepsilon_p + \varepsilon_q)\gamma_A$ , and the corresponding term is valid at  $m \ll \varepsilon_{p,q} \ll m\gamma_B$ . Performing the integration over  $\varepsilon_{p,q}$  in the regions indicated, one has

$$\sigma_B^c = -\frac{28}{9\pi m^2} (Z_A \alpha)^2 (Z_B \alpha)^2 f(Z_B \alpha) \ln^2(\gamma_A \gamma_B). \quad (13)$$



The formulas (12) and (13) can be easily obtained in the Weizsäcker-Williams approximation using the well-known result for the exact in  $Z\alpha$  pair photoproduction cross section in the field of a nucleus. They coincide with the result of [5] (see also [8]). It also follows from the Weizsäcker-Williams method that the contribution of the terms, containing the higher orders of  $Z_A$  and  $Z_B$  simultaneously, can be neglected within our accuracy.

In papers [1, 2, 3] the amplitude of  $e^+e^-$  pair production was obtained at fixed impact parameter between the nuclei. Using this amplitude, it is possible to represent the Coulomb corrections related to the nucleus  $B$  as the integral over the impact parameter:

$$d\sigma_B^c = \frac{m^2 d^3 p d^3 q}{(2\pi)^6 \varepsilon_p \varepsilon_q} \int d^2 \rho \iint \frac{d^2 k_1}{(2\pi)^2} \frac{d^2 k_2}{(2\pi)^2} \exp[i(\mathbf{k}_1 - \mathbf{k}_2)\boldsymbol{\rho}] \mathcal{M}(\mathbf{k}_1) \mathcal{M}^*(\mathbf{k}_2) \times \quad (14)$$

$\times [\mathcal{F}_B(\mathbf{k}_1) \mathcal{F}_B^*(\mathbf{k}_2) - \mathcal{F}_B^0(\mathbf{k}_1) \mathcal{F}_B^{0*}(\mathbf{k}_2)] \mathcal{F}_A^0(\mathbf{q}_\perp + \mathbf{p}_\perp - \mathbf{k}_1) \mathcal{F}_A^{0*}(\mathbf{q}_\perp + \mathbf{p}_\perp - \mathbf{k}_2)$ . Again, changing the order of integration would lead to zero result. Indeed, taking the integral over  $\boldsymbol{\rho}$  first, we get the factor  $\delta(\mathbf{k}_1 - \mathbf{k}_2)$  in the integrand, and, therefore, the integral over  $\mathbf{k}_1$  vanishes due to the relation  $|\mathcal{F}_B|^2 = |\mathcal{F}_B^0|^2$ . Let us demonstrate that, similar to the case of the integral (4) calculation, the change of the integration order in (14) is incorrect, and the result (11) also follows from (14). For this purpose, we restrict the region of integration over  $\boldsymbol{\rho}$  by the condition  $\rho < R$ . After that it is possible to change the order of integration and take the integral over  $\boldsymbol{\rho}$ . Then the main contribution to the integral over  $\mathbf{k}_{1,2}$  comes from the region  $k_{1,2} \leq 1/R$ . Since we are going to take the limit  $R \rightarrow \infty$ , we can replace  $\mathcal{M}(\mathbf{k}_{1,2})$  with  $\mathbf{k}_{1,2}\mathbf{L}$  and neglect  $\mathbf{k}_{1,2}$  in  $\mathcal{F}_A^0(\mathbf{q}_\perp + \mathbf{p}_\perp - \mathbf{k}_{1,2})$ . Then, we have

$$d\sigma_B^c = \frac{m^2 d^3 p d^3 q}{(2\pi)^6 \varepsilon_p \varepsilon_q} |\mathcal{F}_A^0(\mathbf{q}_\perp + \mathbf{p}_\perp)|^2 \frac{|\mathbf{L}|^2}{2} \tilde{G}_B, \quad (15)$$

$$\tilde{G}_B = 8\pi (Z_B \alpha)^2 \int_0^\infty \int_0^\infty \frac{dk_1 dk_2}{k_1^2 - k_2^2} \left\{ \left( \frac{k_1}{k_2} \right)^{2iZ_B \alpha} - 1 \right\} \times$$

$$\times [k_2 R J_0(k_2 R) J_1(k_1 R) - k_1 R J_0(k_1 R) J_1(k_2 R)] .$$

Comparing the expression for the function  $\tilde{G}_B$  with (8), we see that  $\tilde{G}_B = G_B$ . After the summation over the electron and positron polarizations the formula (15) comes into (11). Note that the expression (11) can be obtained directly from (14) by taking the integral over  $\mathbf{k}_{1,2}$  in the region  $k_{1,2} < k_0 \ll |\mathbf{p}_\perp + \mathbf{q}_\perp|$  and then integrating over  $\rho$  in the infinite limits.

If the impact parameter  $\rho$  is restricted by the beam transverse size  $R_0$ , then it follows from the above consideration that the effect of the finite size appears when we can not neglect  $k \sim 1/R_0$  in the argument of  $\mathcal{F}_A^0$  in comparison with  $|\mathbf{q}_\perp + \mathbf{p}_\perp|$ . This is equivalent to the condition  $R_0 \ll 1/\Delta_{min} \sim \gamma_A \gamma_B / m$ .

Thus, the method developed in [1, 2, 3] can be used for the calculation of the Coulomb corrections to the  $e^+e^-$  pair production cross section integrated over the direction of the positron (electron) momentum. Its careful application leads to the correct result.

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