Siberian Branch of Russian Academy of Science

BUDKER INSTITUTE OF NUCLEAR PHYSICS

A.V. Bogdan, V.S. Fadin

A PROOF OF THE REGGEIZED FORM
OF AMPLITUDES WITH QUARK EXCHANGES

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A proof of the reggeized form
of amplitudes with quark exchanges[1]

A.V. Bogdan[†], V.S. Fadin[‡]

[‡] Budker Institute of Nuclear Physics

[†] and Novosibirsk State University

630090 Novosibirsk, Russia

Abstract

A complete proof of the quark Reggeization hypothesis in the leading logarithmic approximation for any quark-gluon inelastic process in the multi-Regge kinematics in all orders of α_s is given. First, we show that the multi-Regge form of QCD amplitudes is guarantied if a set of conditions on the Reggeon vertices and the trajectories is fulfilled. Then, we examine these conditions and show that they are satisfied.

[†]*email address:*   A.V.Bogdan@inp.nsk.su
[‡]*email address:*   Fadin@inp.nsk.su

1   Introduction

Along with the Pomeron, which appears in QCD as a compound state of two Reggeized gluons [1], the hadron phenomenology requires Reggeons, which can be constructed as colorless states of Reggeized quarks and antiquarks. It demands further development of the theory of quark Reggeization [2] in QCD. Till now, this theory remains less developed than the Reggeized gluon theory, although a noticeable progress was achieved in the last years, in particular, the multi–particle Reggeon vertices required in the next–to–leading approximation (NLA) were found [3], and the next–to–leading order (NLO) corrections to the vertices appearing in the the leading logarithmic approximation (LLA) were calculated [4, 5]. All these calculations were performed assuming the quark Reggeization hypothesis. However, this hypothesis was not proved even in the LLA, where merely its self–consistency was shown, in all orders of α_s, but only in a particular case of elastic quark–gluon scattering [2]. Recently, the hypothesis was tested at the NLO in order α_s^2 in [6], where its compatibility with high-energy behaviour of the two–loop quark–gluon scattering amplitude was shown and the NLO correction to the quark trajectory was found in the limit of the space–time dimension $D \to 4$. Then, by the explicit two–loop calculations with the help of s–channel unitarity [7] the hypothesis was checked and corresponding correction to the quark trajectory was found at arbitrary D.

In this paper we suggest a complete proof of the quark Reggeization hypothesis in the LLA for any quark–gluon inelastic process in all orders of α_s. The proof is based on the relations required by compatibility of the multi–Regge form of QCD amplitudes with the s–channel unitarity (bootstrap relations). We derive these relations and show that their fulfilment guaranties the multi–Regge form. Fulfilment of bootstrap relations is secured by several conditions (bootstrap conditions) on the Reggeon vertices and trajectories. We explicitly show that these conditions are satisfied by the known expressions for the vertices and trajectories. The method of the proof is similar to one used for proving of the gluon Reggeization in the NLA [8], but instead of passing to partial waves we apply recently introduced operator formalism [9] extended to consideration of inelastic amplitudes and quark exchanges.

The paper is organized as follows. In the next Section necessary denotations, kinematics definition as well as the form of the multi–Regge inelastic amplitudes are introduced and explicit expressions for particle–particle–Reggeon, Reggeon–Reggeon–particle vertices and quark and gluon trajectories are given. The bootstrap relations are derived in Section 3. Section 4 is devoted to calculation of the s–channel discontinuities of the amplitudes. The bootstrap conditions for Reggeon vertices and trajectory are derived in Section 5. In the subsequent Section 6 these bootstrap conditions are verified. Section 7 concludes the paper.
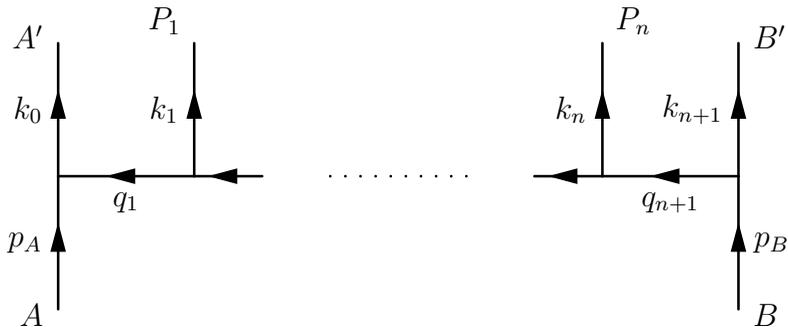
2 The multi–Regge form of QCD amplitudes

Figure 1: Schematic representation of the process $A + B \to A' + P_1 + \cdots + P_n + B'$.

The only kinematics which is essential in the LLA is the multi–Regge kinematics (MRK) which means that all particles participating in a high–energy process are well separated in the rapidity space and have limited transverse momenta.

Let us consider the process $A + B \to A' + P_1 + \ldots + P_n + B'$ in the MRK. We will use light-cone momenta n_1 and n_2, $n_1^2 = n_2^2 = 0$, $(n_1 n_2) = 1$, and denote $(pn_2) \equiv p^+$, $(pn_1) \equiv p^-$, so that $pq = p^+ q^- + p^- q^+ + p_\perp q_\perp$, where the sign $\perp$ means transverse to the (n_1, n_2) plane components. We assume that initial momenta p_A and p_B (see Fig. 1 for denotations) have predominant components along n_1 and n_2 respectively. For generality we do not assume that transverse components $p_{A\perp}$ and $p_{B\perp}$ are zero, but $|p_{A\perp}^2| \sim |p_{B\perp}^2| \sim p_A^2 \sim p_B^2 \ll p_A^+ p_B^-$ and remain limited (do not grow) at $p_A^+ p_B^- \to \infty$.

For the final particle momenta k_i, $i = 0, ..., n+1$, we assume the MRK conditions:

$$k_0^- \ll k_1^- \ll \ldots \ll k_n^- \ll k_{n+1}^-\,,$$
$$k_{n+1}^+ \ll k_n^+ \ll \ldots \ll k_1^+ \ll k_0^+\,, \tag{1}$$

and $k_{i\perp}$ are limited. It ensures that the squared invariant masses $s_{ij} = (k_i + k_j)^2$ are large compared with the squared transverse momenta; at $i < j$

$$s_{ij} \approx 2k_i^+ k_j^- = \frac{k_i^+}{k_j^+}(k_j^2 - k_{j\perp}^2) = \frac{k_j^-}{k_i^-}(k_i^2 - k_{i\perp}^2)\,, \tag{2}$$

and at $i < l < j$ submit to relations

$$s_{il}s_{lj} \approx s_{ij}(k_l^2 - k_{l\perp}^2)\,. \tag{3}$$

For the momentum transfers q_i, $i = 1, \ldots, n+1$,

$$q_1 = k_0 - p_A\,, \ \ q_{j+1} = q_j + k_j\,, \ \ \ (j = 1, \ldots, n)\,, \tag{4}$$

we have

$$q_i^2 \approx q_\perp^2\,. \tag{5}$$

High energy behaviour of amplitudes in the MRK is determined by exchanges in q_i channels. The largest ($\sim s_{AB} \equiv (p_A + p_B)^2$) are amplitudes with gluon exchanges in all channels; such exchanges give factors $s_i \equiv s_{i-1i}$ for each of q_i–channel, and product of all these factors gives s_{AB} due to (3). A quark (antiquark) in a channel with momentum q_j leads to loss of $(s_j)^{1/2}$.

Our goal is to prove that the amplitude $A_{2 \to n+2}$ of the process $A + B \to A' + P_1 + \ldots. + P_n + B'$ has the multi–Regge form

$$A_{2 \to n+2}^\mathcal{R} = \bar{\Gamma}_{A'A}^{\mathcal{R}_1} \frac{s_1^{\omega_1}}{d_1} \gamma_{\mathcal{R}_1 \mathcal{R}_2}^{P_1} \frac{s_2^{\omega_2}}{d_2} \ldots.. \gamma_{\mathcal{R}_i \mathcal{R}_{i+1}}^{P_n} \frac{s_{n+1}^{\omega_{n+1}}}{d_{n+1}} \Gamma_{B'B}^{\mathcal{R}_{n+1}}\,, \tag{6}$$

where $\bar{\Gamma}_{A'A}^\mathcal{R}$ and $\Gamma_{B'B}^\mathcal{R}$ are the particle–particle–Reggeon (PPR) effective vertices, describing $P \to P'$ transitions due to interaction with Reggeons $\mathcal{R}$; for the gluon quantum numbers in q_i channel $\omega_i = \omega_\mathcal{G}(q_i)$ is the gluon Regge trajectory and $d_i \equiv d_i(q_i) = q_{i\perp}^2$; for the quark numbers $\omega_i = \omega_\mathcal{Q}(q_i)$ is the quark Regge trajectory and $d_i \equiv d_i(q_i) = m - \hat{q}_{i\perp}$; $\gamma_{\mathcal{R}_i \mathcal{R}_{i+1}}^{P_i}$ are the Reggeon–Reggeon–particle (RRP) effective vertices, describing production of particles P_i at Reggeon transitions $\mathcal{R}_{i+1} \to \mathcal{R}_i$. For definiteness we do not consider
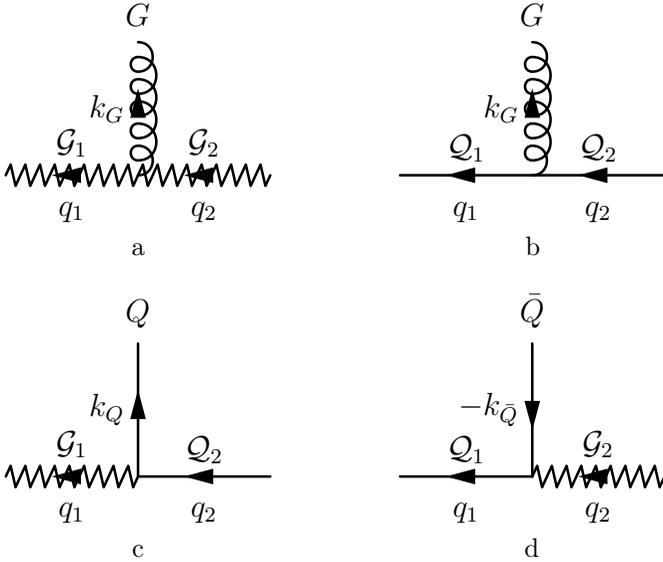
Figure 2: The Reggeon–Reggeon–Particle vertices.

here the antiquark quantum numbers in any of q_i channels. It determines the order of the multipliers in (6). At that, our consideration does not lose generality, since amplitudes with quark and antiquark exchanges are related by charge conjugation.

In order to perform consideration of processes with gluon and quark exchanges in an unified way we introduced in (6) denotations slightly different from usually used. We denote particles and Reggeons by symbols which accumulate all their quantum numbers. We will use the letter P for particles and the letter $\mathcal{R}$ for Reggeons independently of their nature, the letters G and Q for ordinary gluons and quarks and $\mathcal{G}$ and $\mathcal{Q}$ for Reggeized ones. In these denotations we have for the PPR vertices

$$\Gamma^{\mathcal{G}}_{G'G} = -2g\, p_G^- T^{\mathcal{G}}_{G'G}(e^*_{G'_\perp} e_{G_\perp}),\ \Gamma^{\mathcal{G}}_{Q'Q} = g\, \bar{u}_{Q'} t^{\mathcal{G}} \gamma^- u_Q,\ \Gamma^{\mathcal{G}}_{\bar{Q}'\bar{Q}} = -g\, \bar{v}_{\bar{Q}} t^{\mathcal{G}} \gamma^- v_{\bar{Q}'}\, ,$$
$$\bar{\Gamma}^{\mathcal{G}}_{G'G} = -2g\, p_G^+ T^{\mathcal{G}}_{G'G}(e^*_{G'_\perp} e_{G_\perp}),\ \bar{\Gamma}^{\mathcal{G}}_{Q'Q} = g\, \bar{u}_{Q'} t^{\mathcal{G}} \gamma^+ u_Q,\ \bar{\Gamma}^{\mathcal{G}}_{\bar{Q}'\bar{Q}} = -g\, \bar{v}_{\bar{Q}} t^{\mathcal{G}} \gamma^+ v_{\bar{Q}'}\, ,$$
$$\tag{7}$$

$$\Gamma^{\mathcal{Q}}_{G'Q} = -gt^{G'}\hat{e}^*_{G'\perp}u_Q\,, \quad \Gamma^{\mathcal{Q}}_{\bar{Q}'G} = -gt^G\hat{e}_{G\perp}v_{\bar{Q}'}\,,$$

$$\bar{\Gamma}^{\mathcal{Q}}_{G'\bar{Q}} = -g\bar{v}_{\bar{Q}}t^{G'}\hat{e}^*_{G'\perp}\,, \quad \bar{\Gamma}^{\mathcal{Q}}_{Q'G} = -g\bar{u}_{Q'}t^G\hat{e}_{G\perp}\,. \tag{8}$$

As usually, we do not write colour and spinor quark indices; $T^{\mathcal{G}}$ and t^G are the color group generators in the adjoint and fundamental representations. Here and in the following the physical light–cone gauges

$$(e_P k_P) = (e_P n_2) = 0\,, \quad e_P = e_{P\perp} - \frac{(e_{P\perp}k_P)}{k_P^+}n_2 \tag{9}$$

and

$$(e_P k_P) = (e_P n_1) = 0\,, \quad e_P = e_{P\perp} - \frac{(e_{P\perp}k_P)}{k_P^-}n_1 \tag{10}$$

are assumed for polarization vectors e_P of particles P having momenta k_P with predominant components along n_1 and n_2 respectively.

For production of gluon with momentum $k_G = q_2 - q_1$ and polarization vector e in transition $\mathcal{R}_2 \to \mathcal{R}_1$ of Reggeons with momenta q_2 and q_1 we have [10] for the case of Reggeized gluons in both channels (see Fig. 2a)

$$\gamma^G_{\mathcal{G}_1\mathcal{G}_2} = -gT^G_{\mathcal{G}_1\mathcal{G}_2}\,e^{*\mu}C_\mu(q_2,q_1)\,,$$

$$C^\mu(q_2,q_1) = -(q_1+q_2)^\mu_\perp - n^\mu_1\left(k_G^+ + \frac{q^2_{1\perp}}{k_G^-}\right) + n^\mu_2\left(k_G^- + \frac{q^2_{2\perp}}{k_G^+}\right)\,, \tag{11}$$

and (see Fig. 2b))

$$\gamma^G_{\mathcal{Q}_1\mathcal{Q}_2} = -g\,t^G\,e^{*\mu}\mathcal{P}_\mu(q_2,q_1)\,,$$

$$\mathcal{P}^\mu(q_2,q_1) = \gamma^\mu_\perp - (m-\hat{q}_{1\perp})\frac{n^\mu_1}{k_G^-} + (m-\hat{q}_{2\perp})\frac{n^\mu_2}{k_G^+} \tag{12}$$

in the case of Reggeized quarks [2]. It is easy to check, that these vertices are gauge invariant, since

$$C^\mu(q_2,q_1)k_{G\mu} = \mathcal{P}^\mu(q_2,q_1)k_{G\mu} = 0\,. \tag{13}$$

In the gauges (9) and (10) the vertices can be presented as

$$\gamma^G_{\mathcal{G}_1\mathcal{G}_2} = 2gT^G_{\mathcal{G}_1\mathcal{G}_2}\,e^*_\perp\left(q_{1\perp} + k_{G\perp}\frac{q^2_{1\perp}}{k^2_{G\perp}}\right)\,,$$

$$\gamma^G_{\mathcal{Q}_1\mathcal{Q}_2} = -gt^G\,e^*_\perp\left(\gamma_\perp - 2(m-\hat{q}_{1\perp})\frac{k_{G\perp}}{k^2_{G\perp}}\right)\,, \tag{14}$$

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and

$$\gamma^G_{\mathcal{G}_1 \mathcal{G}_2} = 2g T^G_{\mathcal{G}_1 \mathcal{G}_2} \, e^*_\perp \left(q_{2\perp} - k_{G\perp} \frac{q^2_{2\perp}}{k^2_{G\perp}} \right) ,$$

$$\gamma^G_{\mathcal{Q}_1 \mathcal{Q}_2} = -g t^G \, e^*_\perp \left(\gamma_\perp + 2(m - \hat{q}_{2\perp}) \frac{k_{G\perp}}{k^2_{G\perp}} \right) \tag{15}$$

respectively.

The vertices for quark (antiquark) production were found in [2]. For the case of Reggeized gluon in the q_1 channel (see Fig. 2c) we have

$$\gamma^Q_{\mathcal{G}_1 \mathcal{Q}_2} = g \, \bar{u}_Q \frac{\hat{q}_{1\perp}}{k^+_Q} t^{\mathcal{G}_1} , \tag{16}$$

and in the q_2 channel (see Fig. 2d)

$$\gamma^{\bar{Q}}_{\mathcal{Q}_1 \mathcal{G}_2} = -g t^{\mathcal{G}_2} \frac{\hat{q}_{2\perp}}{k^-_{\bar{Q}}} v_{\bar{Q}} . \tag{17}$$

In terms of integrals in the transverse momentum space the Reggeon trajectories are presented as

$$\omega_{\mathcal{G}}(q) = \frac{N_c}{2} \frac{g^2 q^2}{(2\pi)^{D-1}} \int \frac{\mathrm{d}^{D-2} k_\perp}{k^2_\perp (q-k)^2_\perp} ,$$

$$\omega_{\mathcal{Q}}(q_i) = C_F \frac{g^2}{(2\pi)^{D-1}} (m - \hat{q}_\perp) \int \frac{\mathrm{d}^{D-2} k_\perp}{(m - \hat{k}_\perp)(q-k)^2_\perp} , \tag{18}$$

where $N_c = 3$ for QCD is number of colours, $D = 4 + 2\epsilon$ is the space–time dimension taken different from 4 to regularize infrared divergences.

In the following we will need more general multi–particle amplitudes $A^{\mathcal{R}}_{2+n_1 \to 2+n_2}$, but in the same multi–Regge kinematics. Assuming the same ordering in longitudinal components, the amplitudes $A^{\mathcal{R}}_{2+m \to 2+n-m}$ can be obtained from $A^{\mathcal{R}}_{2 \to n+2}$ by usual crossing rules. Note that in (6) we neglect imaginary parts of the amplitude since they are subleading. Therefore the crossing rules for the transition to the amplitudes do not affect the Regge factors $s^{\omega_i}_i$.

Here it seems sensible to make two remarks. The first one is that the hypothesis (6) means much more than it is usually included in the notion "Reggeization" of elementary particles. It means not only existence of the Reggeons with gluon and quark quantum numbers and trajectories (18), but also that in the LLA all the MRK amplitudes are determined only by the

Reggeon exchanges, i.e. only amplitudes with Reggeon quantum numbers (that means, in particular, colour octet for pure gluon exchanges and colour triplet for exchanges with flavour) do survive. The second remark concerns signature. As compared with ordinary particles Reggeons possess additional quantum numbers – signature, negative for the Reggeized gluon and positive for the Reggeized quark. Therefore, in order to affirm that the amplitude is given by the Reggeon exchanges we need to show that it has corresponding signatures in all q_i–channels.

In order to construct amplitudes with definite signatures one needs to perform "signaturization". In general the signaturization is not a simple task. It requires partial-wave decomposition of amplitudes in cross-channels with subsequent symmetrization (anti–symmetrization) in "scattering angles" and analytical continuation into the s–channel. The procedure is relatively simple only in the case of elastic scattering of spin-zero particles. At that, generally speaking, even in this case the amplitudes with definite signatures can not be expressed in terms of physical amplitudes related by crossing. Fortunately, at high energy the signaturization can be easily done not only for elastic, but in the MRK also for inelastic amplitudes, for particles with spin as well as for spin–zero ones. The signaturization (as well as crossing relations) is naturally formulated for "truncated" amplitudes, i.e. for amplitudes with omitted wave functions (polarization vectors and Dirac spinors). The crucial points are that in the MRK all energy invariants s_{ij} are large and that they are determined only by longitudinal components of momenta ($s_{ij} = 2p_i^+ p_j^-$, $i < j$). Due to largeness of s_{ij} signaturization in the q_l–channel means symmetrization (anti–symmetrization) with respect to the substitution $s_{ij} \leftrightarrow -s_{ij}$, $i < l \leq j$. Since s_{ij} are determined by longitudinal components, it can be considered as the substitution $k_i^\pm \leftrightarrow -k_i^\pm$, $i < l$, $p_A^\pm \leftrightarrow -p_A^\pm$ (or, equivalently, $k_j^\pm \leftrightarrow -k_j^\pm$, $j \geq l$, $p_B^\pm \leftrightarrow -p_B^\pm$) in truncated amplitudes without change of transverse components. Note that such substitution does not violate momentum conservation due to strong ordering of the longitudinal components (1). At that, all particles remain on their mass shell, so that the substitution is equivalent to transition into the cross-channel.

In order to understand behaviour of the amplitudes (6) under the signaturization it is convenient to take the gluon production vertices in the physical light–cone gauges with gauge–fixing vectors n_2 or n_1 (see (14), (15)). At that, it becomes evident that they do not depend on longitudinal components of momenta, as well as the PPR vertices for the Reggeized quark (8) after omitting of wave functions. On the contrary, the quark and antiquark production vertices (16) and (17) contain explicitly longitudinal components, so that they change their signs at the transition into the cross-channel. The same is true

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for the PPR vertices with the Reggeized gluon (7): for the vertices for gluon scattering because they are proportional to longitudinal components, and for the vertices for quark and antiquark scattering because of difference in their signs. After these remarks, with account of the fact that in the LLA change of signs of s_i does not affect the Regge factors, it is not difficult to see that the amplitudes (6) are invariant with respect to the signaturization described above, i.e. they have corresponding signatures in each of the q_i–channels.

3   Bootstrap relations

The proof of the form (6) is based on use of the s–channel unitarity, which provides us with discontinuities $disc_{s_{ij}}$ (i.e. imaginary parts) of the amplitudes in the s_{ij} channels. We need to connect the amplitudes themselves (which are real in the LLA) with these discontinuities. It is not difficult to do for elastic amplitudes. Unfortunately, it is quite not so for inelastic amplitudes. Analytical properties of the production amplitudes are very complicated even in the MRK [11]. But fortunately, it turns out, that in the LLA these properties are greatly simplified and allow us to express partial derivatives $\partial/\partial \ln(s_i)$ of the amplitudes, considered as a function of s_i, $i = 1 \ldots n+1$, and transverse momenta, in terms of the discontinuities of the signaturized amplitudes. It permits us to find all the MRK amplitudes loop by loop in the perturbation theory, using the Born form of these amplitudes and the unitarity relations. Note that in the Born approximation the representation (6) was proved in [1, 2] with the help of the t–channel unitarity.

For the elastic amplitude the partial derivative $\partial/\partial \ln s$ can be expressed in terms of the s–channel discontinuity quite easily. For the signaturized amplitudes radiative corrections depend on s only in the form $(\ln^n(-s) + \ln^n s)$ independently of signature. With the LLA accuracy we can put

$$\frac{1}{-\pi \mathrm{i}} \mathrm{disc}_s \left(\ln^n(-s) + \ln^n s \right) = \frac{\partial}{\partial \ln s} \left[\ln^n(-s) + \ln^n s \right] . \qquad (19)$$

Therefore we have (the superscript $\mathcal{S}$ means signaturization)

$$\frac{1}{-\pi \mathrm{i}} \mathrm{disc}_s \left[A_{2 \to 2}^{\mathcal{S}} \right] / A_{2 \to 2}^{\mathrm{Born}} = \frac{\partial}{\partial \ln s} \left[A_{2 \to 2}^{\mathcal{S}} / A_{2 \to 2}^{\mathrm{Born}} \right] . \qquad (20)$$

Division by the Born amplitude is performed in order to differentiate s–dependence of radiative corrections only.

In the case of $A_{2 \to 2+n}$ the main complication is that instead of s we have $(n+2)(n+1)/2$ large invariants $s_{ij} = (k_i + k_j)^2$, which are not independent

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because of the equalities (3). Equalities like (20) connecting discontinuities in each of the channels and corresponding derivatives of the amplitude do not exist. However there are equalities [12] connecting definite combinations of the discontinuities and the derivatives $\partial/\partial s_i$:

$$
\frac{1}{-\pi \mathrm{i}} \left(\sum_{l=k+1}^{n+1} \mathrm{disc}_{s_{kl}} - \sum_{l=0}^{k-1} \mathrm{disc}_{s_{lk}} \right) A_{2 \to n+2}^{\mathcal{S}} / A_{2 \to n+2}^{\mathrm{Born}} =
$$
$$
\left(\frac{\partial}{\partial \ln s_{k+1}} - \frac{\partial}{\partial \ln s_k} \right) \left[A_{2 \to n+2}^{\mathcal{S}}(s_i) / A_{2 \to n+2}^{\mathrm{Born}} \right] .
\tag{21}
$$

Here in the r.h.s. the amplitude is expressed in terms of s_i, $i = 1 \ldots n+1$, and transverse momenta; the index k takes values from 0 to $n+1$.

Equalities (21) can be easily proved with use of the Steinmann relations, or, more definitely, of the statement [11] that the amplitude can be presented as a sum of contributions corresponding to various sets of $n+1$ nonoverlapping channels $s_{i_k j_k}$, $i_k < j_k$, $k = 1 \ldots n+1$; at that each of the contributions can be written as a signaturized series in logarithms of energy variables $s_{i_k j_k}$ with coefficients which are real function of transverse momenta. Remind that two channels $s_{i_1 j_1}$ and $s_{i_2 j_2}$ are called overlapping if either $i_1 < i_2 \le j_1 < j_2$, or $i_2 < i_1 \le j_2 < j_1$. What is important:

— energy variables $s_{i_k j_k}$ are independent, since the relations (3) concern with overlapping channels; it means, in particular, that we need to consider only leading orders in logarithms of these variables;

— we need not to consider the coefficients depending on transverse momenta neither calculating the discontinuities, nor calculating derivatives over $\ln s_i$.

Therefore, since scattering amplitudes enter the relations (21) linearly and uniformly, it is sufficient to prove these relations in the leading order for the symmetrized products

$$
SP = \hat{\mathcal{S}} \prod_{i<j=1}^{n+1} (-s_{ij})^{\alpha_{ij}}
\tag{22}
$$

instead of $A_{2 \to 2+n}^{\mathcal{S}} / A_{2 \to 2+n}^{\mathrm{Born}}$. Here the exponents $\alpha_{ij} \sim g^2$ are different from zero only for some set of nonoverlapping channels and are arbitrary in all other respects; $\hat{\mathcal{S}}$ means symmetrization with respect to simultaneous change of signs of all s_{ij} with $i < k \le j$, performed independently for each $k = 1 \ldots n+1$. Indeed, due to above mentioned arbitrariness of α_{ij} fulfilment of (21) for SP guarantees it for any logarithmic series.

With $\alpha_{ij} \sim g^2$ calculating discontinuity of SP in one of the invariants s_{ij} we can neglect in the leading order signs of other invariants, so that we have

$$\frac{1}{-\pi i}\left(\sum_{l=k+1}^{n+1} \text{disc}_{s_{kl}} - \sum_{l=0}^{k-1}\text{disc}_{s_{lk}}\right)SP = \left(\sum_{l=k+1}^{n+1}\alpha_{kl} - \sum_{l=0}^{k-1}\alpha_{lk}\right)SP. \quad (23)$$

From other hand, taking into account that with the LO accuracy,

$$(s_{ij})^{\alpha_{ij}} = \prod_{l=i+1}^{j} s_l^{\alpha_{ij}}, \quad (24)$$

we have

$$\left(\frac{\partial}{\partial \ln s_{k+1}} - \frac{\partial}{\partial \ln s_k}\right)SP = \left(\sum_{i<k+1, j\geq k+1}\alpha_{ij} - \sum_{i<k, j\geq k}\alpha_{ij}\right)SP$$

$$= \left(\sum_{l=k+1}^{n+1}\alpha_{kl} - \sum_{l=0}^{k-1}\alpha_{lk}\right)SP. \quad (25)$$

From (23) and (25) it follows that the equalities (21) are fulfilled.

These equalities allow us to express all partial derivatives $\partial/\partial\ln(s_k)A_{2\to 2+n}$ through the discontinuities. Note that from $n+2$ equalities (21) considered as equations for the derivatives only $n+1$ are linear independent, that can be easily seen taking sum of the equations over $k = 0\ldots n+1$. Note here that requirement of equality of mixed derivatives taking in different orders imposes strong restrictions on the discontinuities. If they are fulfilled, the amplitude is unambiguously defined by its value at $\ln s_i = 0$, i.e. in the Born approximation. It means that the equalities (21) permit to find in the LLA all the MRK amplitudes using the Born approximation for them and the s–channel unitarity. Indeed, at some number L of loops the discontinuities entering (21) can be expressed with the help of the s–channel unitarity through the amplitudes with smaller number of loops. Therefore starting with the expression (6) in the Born approximation (as it was already mentioned, in this approximation it was proved for arbitrary n [1, 2] with the help of the t–channel unitarity) we can calculate loop–by–loop all radiative corrections to the Born amplitudes and examine the formula (6).

Instead of such calculations it is sufficient, since the amplitudes are determined unambiguously, to check that the Reggeized form (6) satisfies (21).

Substituting (6) into the r.h.s. of (21) we obtain the bootstrap relations:

$$
\frac{1}{-\pi \mathrm{i}} \left(\sum_{l=k+1}^{n+1} \mathrm{disc}_{s_{kl}} - \sum_{l=0}^{k-1} \mathrm{disc}_{s_{lk}} \right) A^{\mathcal{S}}_{2 \to 2+n}
$$

$$
= \bar{\Gamma}^{\mathcal{R}_1}_{A'A} \frac{s_1^{\omega_1}}{d_1} \prod_{i=2}^{k} \left(\gamma^{P_{i-1}}_{\mathcal{R}_{i-1}\mathcal{R}_i} \frac{s_i^{\omega_i}}{d_i} \right) \left[\gamma^{P_k}_{\mathcal{R}_k \mathcal{R}_{k+1}} \omega_{k+1} - \omega_k \gamma^{P_k}_{\mathcal{R}_k \mathcal{R}_{k+1}} \right] \tag{26}
$$

$$
\times \frac{s_{k+1}^{\omega_{k+1}}}{d_{k+1}} \prod_{i=k+2}^{n+1} \left(\gamma^{P_{i-1}}_{\mathcal{R}_{i-1}\mathcal{R}_i} \frac{s_i^{\omega_i}}{d_i} \right) \Gamma^{\mathcal{R}_{n+1}}_{B'B} .
$$

In the l.h.s. of these equations the discontinuities must be calculated using the unitarity relations and the anzats (6). Since number of the bootstrap relations is infinite it is quite nontrivial to satisfy all of them using only several Reggeon vertices and trajectories. A crucial for the Reggeization hypothesis fact, which is demonstrated below, is that all these relations are fulfilled if the Reggeized vertices and trajectories satisfy several equations called bootstrap conditions. In the following we derive these conditions and demonstrate that they are satisfied.

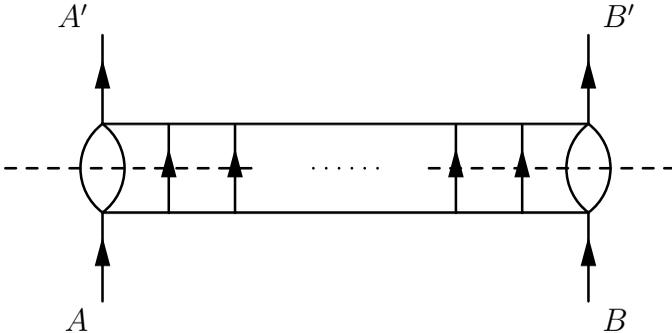# 4   Calculation of the discontinuities

Figure 3:   Schematic representation of the s–channel discontinuity $\mathrm{disc}_s A^{\mathcal{S}}_{AB \to A'B'}$.

Let us start with the elastic amplitude. For the process $A + B \to A' + B'$

the discontinuity is

$$\text{disc}_s A_{AB \to A'B'}^{\mathcal{S}} = \mathrm{i} \sum_{n=0}^{\infty} \hat{\mathcal{S}} \int A_{AB \to n+2}^{\mathcal{R}} A_{A'B' \to n+2}^{\mathcal{R}} \mathrm{d}\rho_{n+2}, \qquad (27)$$

where $\hat{\mathcal{S}}$ is the signaturization operator, the sum is taken over discrete quantum states of intermediate particles as well as over their number, $\mathrm{d}\rho_{n+2}$ is their phase–space element, and the hermicity property of the amplitudes (6) is used. The discontinuity is presented schematically at Fig. 3, where the circles on the lines AA' and BB' mean the signaturization (evidently its execution for both lines gives the same result as for one of them).

To calculate the discontinuity we need to convolute Reggeon vertices and to integrate over momenta of particles in the intermediate states. All convolutions are known long ago [1, 2]. The important fact is that they do not depend on longitudinal momenta. In order to present them, and then the discontinuities, in a compact way it is convenient to use operator denotations in the transverse momentum, colour and spin space. We will use also denotations which accumulate all these quantum numbers. Thus, $\langle \mathcal{G}_i|$ and $|\mathcal{G}_i\rangle$ are "bra"– and "ket"–vectors for the t– channel states of the Reggeized gluon with transverse momentum $r_{i\perp}$ and colour index c_i. It is convenient to define the scalar product

$$\langle \mathcal{G}_i | \mathcal{G}_j \rangle = r_{i\perp}^2 \delta(r_{i\perp} - r_{j\perp}) \delta_{c_i c_j}. \qquad (28)$$

Analogously, $\langle \mathcal{Q}_i|$ and $|\mathcal{Q}_i\rangle$ with the scalar product

$$\langle \mathcal{Q}_i | \mathcal{Q}_j \rangle = (m - \hat{r}_{i\perp})_{\rho_i \rho_j} \delta(r_{i\perp} - r_{j\perp}) \delta_{\alpha_i \alpha_j} \qquad (29)$$

denote the t– channel states of the Reggeized quark with transverse momentum $r_{i\perp}$, colour index α_i and spinor index ρ_i. We will use the letter $\mathcal{R}$ for denotation of Reggeon states independently of their nature. In the following we will use the letters $\mathcal{G}_i$ and $\mathcal{Q}_i$ also as colour indices, instead of c_i and α_i. The states with two Reggeons are built from the above ones. At that it is convenient to distinguish the states $|\mathcal{R}_i \mathcal{R}_j\rangle$ (with corresponding "bra"– vectors $\langle \mathcal{R}_i \mathcal{R}_j|$)and $|\mathcal{R}_j \mathcal{R}_i\rangle$. We will associate the first of them with the case when the Reggeon $\mathcal{R}_i$ turns up in the lower part of Fig. 3, i.e. in the amplitude $A_{AB \to n+2}^{\mathcal{R}}$, and the second with the case when it turns up in the upper part of Fig. 3, i.e. in the amplitude $A_{n+2 \to A'B'}^{\mathcal{R}}$. We define three types of states

$$|\mathcal{G}_i \mathcal{G}_j\rangle = |\mathcal{G}_i\rangle |\mathcal{G}_j\rangle, \ \ |\mathcal{G}_i \mathcal{Q}_j\rangle = |\mathcal{G}_i\rangle |\mathcal{Q}_j\rangle, \ \ |\mathcal{Q}_i \mathcal{G}_j\rangle = |\mathcal{Q}_i\rangle |\mathcal{G}_j\rangle. \qquad (30)$$

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States of different types are orthogonal one another. All states create a complete set, i.e.

$$\langle\Psi|\Phi\rangle = \int \langle\Psi|\mathcal{G}_1\mathcal{G}_2\rangle \frac{\mathrm{d}^{D-2}r_{1\perp}\mathrm{d}^{D-2}r_{2\perp}}{r_{1\perp}^2 r_{2\perp}^2} \langle\mathcal{G}_1\mathcal{G}_2|\Phi\rangle$$

$$+ \int \langle\Psi|\mathcal{Q}_1\mathcal{G}_2\rangle \frac{\mathrm{d}^{D-2}r_{1\perp}\mathrm{d}^{D-2}r_{2\perp}}{(m-\hat{r}_{1\perp})r_{2\perp}^2} \langle\mathcal{Q}_1\mathcal{G}_2|\Phi\rangle$$

$$+ \int \langle\Psi|\mathcal{G}_1\mathcal{Q}_2\rangle \frac{\mathrm{d}^{D-2}r_{1\perp}\mathrm{d}^{D-2}r_{2\perp}}{(m-\hat{r}_{2\perp})r_{1\perp}^2} \langle\mathcal{G}_1\mathcal{Q}_2|\Phi\rangle, \qquad (31)$$

where summation over colour and spin indices is assumed.

Interaction of scattering particles with Reggeons is described by so called impact factors. We define them as projections of t-channel states $|\bar{B}'B\rangle$ and $\langle A'\bar{A}|$ on the two-Reggeon states:

$$\langle\mathcal{R}_1\mathcal{R}_2|\bar{B}'B\rangle = \delta(r_{1\perp}+r_{2\perp}-q_{B\perp})\frac{1}{2p_B^-}\sum_P \left(\Gamma_{B'P}^{\mathcal{R}_2}\Gamma_{PB}^{\mathcal{R}_1} \pm \underline{\Gamma}_{\bar{B}P}^{\mathcal{R}_2}\underline{\Gamma}_{PB'}^{\mathcal{R}_1}\right), \quad (32)$$

where the $+$ $(-)$ sign stands for a fermion (boson) state in the t–channel, $q_B = p_B - p_{B'}$, the sum is taken over quantum numbers of particles P (at that, these particles can be different in the first and the second terms) and the factor $1/p_B^-$ is included in the definition for convenience. The factor $\frac{1}{2}$ and the last term in (32) serves for account of the signaturization; at that, the bar over particle symbols means, as usually, antiparticles, while $\underline{\Gamma}_{\bar{B}P}^{\mathcal{R}_2}$ and $\underline{\Gamma}_{PB'}^{\mathcal{R}_1}$ are obtained from $\Gamma_{\bar{B}P}^{\mathcal{R}_2}$ and $\Gamma_{PB'}^{\mathcal{R}_1}$ correspondingly (see (7), (8)) taking instead of wave functions (polarization vectors and Dirac spinors) of $\bar{B}$ and $\bar{B}'$ the wave functions of B and B' from the first term.

Quite analogously,

$$\langle A'\bar{A}|\mathcal{R}_1\mathcal{R}_2\rangle = \delta(r_{1\perp}+r_{2\perp}-q_{A\perp})\frac{1}{2p_A^+}\sum_P \left(\bar{\Gamma}_{A'P}^{\mathcal{R}_2}\bar{\Gamma}_{PA}^{\mathcal{R}_1} \pm \underline{\bar{\Gamma}}_{\bar{A}P}^{\mathcal{R}_2}\underline{\bar{\Gamma}}_{P\bar{A}'}^{\mathcal{R}_1}\right), \quad (33)$$

where $q_A = p_{A'} - p_A$.

We introduce the operator $\hat{\mathcal{K}}_r$ of Reggeon-Reggeon interaction, related to real particle production. It is defined by its matrix elements between the two-Reggeon states, which are expressed in terms of convolutions of the RRP vertices. The important remark which must be made here is that, because of the anticommutativity of the fermion operators the sign of the amplitude depends on their order in the definition of the state vectors. We have defined the amplitudes $A^{\mathcal{R}}$ (6) without worrying about their signs or fixing this order,

as if the operators were commutative. However in (27) the relative signs of the amplitudes must be taken into account. In order to do this we must associate a factor -1 with each antiquark in the intermediate state (that can be easily understood from the Cutkosky rules). We define:

$$\langle \mathcal{R}_1 \mathcal{R}_2 | \hat{\mathcal{K}}_r | \mathcal{R}'_1 \mathcal{R}'_2 \rangle = \delta(q'_\perp - q_\perp) \frac{1}{2(2\pi)^{D-1}} \sum_P \gamma^P_{\mathcal{R}_1 \mathcal{R}'_1} \gamma^{\mathcal{R}_2 \mathcal{R}'_2}_P, \qquad (34)$$

where $q_\perp = r_{1\perp} + r_{2\perp}$, $q'_\perp = r'_{1\perp} + r'_{2\perp}$. In this formula we account the above remark concerning -1 for each antiquark in intermediate state by insertion -1 into the definition of the vertex $\gamma^{\mathcal{G}_2 \mathcal{Q}'_2}_{\bar{Q}}$:

$$\gamma^{\mathcal{G}_2 \mathcal{Q}'_2}_{\bar{Q}} = g\, \bar{v}_Q \, \frac{\hat{p}_{\mathcal{G}_2 \perp}}{k^+_{\bar{Q}}} \, t^{\mathcal{G}_2}, \qquad (35)$$

$$\gamma^{\mathcal{Q}_2 \mathcal{G}'_2}_Q = g\, t^{\mathcal{G}'_2} \frac{\hat{p}_{\mathcal{G}'_2 \perp}}{k^-_Q} \, u_Q; \qquad (36)$$

and the vertices $\gamma^{\mathcal{R}_2 \mathcal{R}'_2}_G$ are obtained from $\gamma^G_{\mathcal{R}_2 \mathcal{R}'_2}$ (see (14)), (15)) by the substitution $k_G \to -k_G$ (in accordance with momentum conservation) and $e^*_G \to e_G$. We introduce also the operator $\hat{\Omega}$, so that

$$\hat{\Omega} |\mathcal{R}_1 \mathcal{R}_2 \rangle = (\omega_{R_1}(r_{1\perp}) + \omega_{R_2}(r_{2\perp})) |\mathcal{R}_1 \mathcal{R}_2 \rangle. \qquad (37)$$

Denoting momenta of intermediate particles by k_i, we have for the phase space element in (27)

$$\mathrm{d}\rho_{n+2} = (2\pi)^D \delta(p_A + p_B - \sum_{i=0}^{n+1} k_i) \prod_{i=0}^{n+1} \frac{\mathrm{d}^{D-1} k_i}{2k_i^0 (2\pi)^{D-1}}$$

$$= \frac{(2\pi)^D}{p_A^+ p_B^-} \delta(p_{A\perp} + p_{B\perp} - \sum_{i=0}^{n+1} k_{i\perp}) \prod_{j=1}^{n} \mathrm{d}y_i \prod_{i=0}^{n+1} \frac{\mathrm{d}^{D-2} k_{i\perp}}{2(2\pi)^{D-1}}, \qquad (38)$$

where $y_i = \ln k_i^+$– rapidities of the produced particles, obeying the conditions

$$\ln p_A^+ \equiv y_A > y_1 > \ldots > y_n > y_B \equiv -\ln p_B^-. \qquad (39)$$

Note that we have included the factors $1/p_B^-$ and $1/p_A^+$ in the definitions of the impact–factors (32) and (33), and the factors $(2(2\pi)^{D-1})^{-1}$ from produced particles P_i in the definition of the matrix elements of the kernel (34). Now,

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taking into account that with the LLA accuracy $s_i^{\omega_i} = \mathrm{e}^{\omega_i(y_{i-1}-y_i)}$, we can present the discontinuity (27) in the form

$$\delta(q_{A\perp} - q_{B\perp})\mathrm{disc}_s A_{AB\to A'B'}^{\mathcal{R}} = \frac{\mathrm{i}}{4(2\pi)^{D-2}} \langle A'\bar{A}|\hat{G}(Y)|\bar{B}'B\rangle\,, \qquad (40)$$

where $q_{B\perp} = p_{B\perp} - p_{B'\perp}$, $\ q_{A\perp} = p_{A'\perp} - p_{A\perp}$, $Y = y_A - y_B$ and

$$\hat{G}(Y) = \sum_{n=0}^{\infty} \left(\int_{y_B}^{y_A} \mathrm{e}^{\hat{\Omega}(y_A - y_1)}\mathrm{d}y_1 \hat{\mathcal{K}}_r \int_{y_B}^{y_1} \mathrm{e}^{\hat{\Omega}(y_1 - y_2)}\mathrm{d}y_2 \hat{\mathcal{K}}_r \right.$$

$$\left. \times \cdots \times \int_{y_B}^{y_{n-1}} \mathrm{e}^{\hat{\Omega}(y_{n-1} - y_n)}\mathrm{d}y_n \hat{\mathcal{K}}_r \mathrm{e}^{\hat{\Omega}(y_n - y_B)} \right)\,. \qquad (41)$$

It is easy to see that the Green–function operator obeys the equation

$$\frac{d\hat{G}(Y)}{dY} = \hat{\mathcal{K}}\hat{G}(Y)\,, \qquad (42)$$

where

$$\hat{\mathcal{K}} = \hat{\Omega} + \hat{\mathcal{K}}_r\,, \qquad (43)$$

with initial condition $\hat{G}(0) = 1$, so that

$$\hat{G}(Y) = \mathrm{e}^{\hat{\mathcal{K}}Y} = s^{\hat{\mathcal{K}}}\,. \qquad (44)$$

Eqs. (40) and (44) give the the operator representation of the discontinuities of elastic amplitudes.
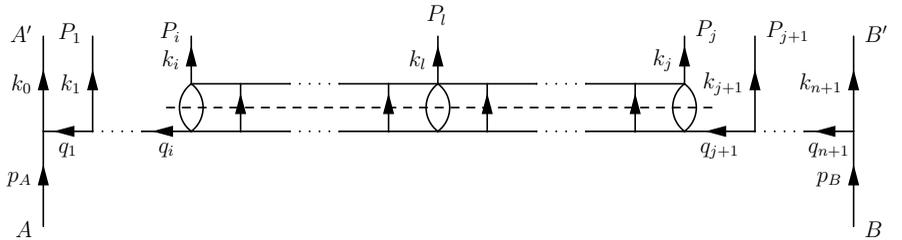

Figure 4: Schematic representation of the s_{ij}–channel discontinuity $\mathrm{disc}_{s_{ij}} A_{2\to n+2}^{\mathcal{S}}$.

To give analogous representations for discontinuities of inelastic amplitudes we need to define new operators and new matrix elements. Let us consider the discontinuity schematically presented at Fig. 4, where the circles, as well as in Fig. 3, mean the signaturization. Analogously to the impact factors for scattering particles we define the impact factors for Reggeon–particle transitions as (compare with (32))

$$\langle \mathcal{R}_1\mathcal{R}_2|\bar{P}_j\mathcal{R}_{j+1}\rangle = \delta(r_{1\perp}+r_{2\perp}-q_{j\perp})\frac{1}{2k_j^-}\sum_P\left(\Gamma^{\mathcal{R}_2}_{\bar{P}_jP}\gamma^P_{\mathcal{R}_1\mathcal{R}_{j+1}} \pm \underline{\Gamma}^{\mathcal{R}_1}_{P\bar{P}_j}\gamma^{\mathcal{R}_2\mathcal{R}_{j+1}}_P\right),$$

(45)

where $q_{j\perp} = q_{(j+1)\perp} - k_{j\perp}$, the $+$ ($-$) sign stands for the case of boson (fermion) production; and

$$\langle P_i\mathcal{R}_i|\mathcal{R}_1\mathcal{R}_2\rangle = \delta(r_{1\perp}+r_{2\perp}-q_{(i+1)\perp})\frac{1}{2k_i^+}\sum_P\left(\bar{\Gamma}^{\mathcal{R}_2}_{\bar{P}_iP}\gamma^P_{\mathcal{R}_i\mathcal{R}_1} \pm \bar{\underline{\Gamma}}^{\mathcal{R}_1}_{P\bar{P}_i}\gamma^{\mathcal{R}_i\mathcal{R}_2}_P\right),$$

(46)

where $q_{(i+1)\perp} = q_{i\perp} + k_{i\perp}$.

Finally, we introduce the operator $\hat{\mathcal{P}}_l$ for production of particle P_l with momentum k_l as having the following matrix elements:

$$\langle \mathcal{R}_1\mathcal{R}_2|\hat{\mathcal{P}}_l|\mathcal{R}_1'\mathcal{R}_2'\rangle = \delta(q_{(l+1)\perp} - k_{l\perp} - q_{l\perp})$$

$$\times \left(\gamma^{P_l}_{\mathcal{R}_1\mathcal{R}_1'}\delta_{\mathcal{R}_2\mathcal{R}_2'}\delta(r_{2\perp} - r_{2\perp}')d_{\mathcal{R}_2} + \gamma^{P_l}_{\mathcal{R}_2\mathcal{R}_2'}\delta_{\mathcal{R}_1\mathcal{R}_1'}\delta(r_{1\perp} - r_{1\perp}')d_{\mathcal{R}_1}\right),$$

(47)

where $q_{l\perp} = r_{1\perp} + r_{2\perp}$, $q_{(l+1)\perp} = r_{1\perp}' + r_{2\perp}'$.

Now we are ready to give the operator representation for discontinuities of the signaturized inelastic amplitudes in the s_{ij}–channels. If $0 < i < j < n + 1$ (see Fig. 4) then the value of $-4\mathrm{i}(2\pi)^{D-2}\delta(q_{i\perp} - q_{(j+1)\perp} - \sum_{l=i}^j k_{l\perp})$ disc$_{s_{ij}} A^{\mathcal{S}}_{2\to n+2}$ can be obtained from the r.h.s of (6) by the replacement of

$$\gamma^{P_i}_{\mathcal{R}_i\mathcal{R}_{i+1}}\left(\prod_{l=i+1}^j \frac{s_l^{\omega_l}}{d_l}\gamma^{P_l}_{\mathcal{R}_l\mathcal{R}_{l+1}}\right) \longrightarrow \langle P_i,\mathcal{R}_i|\left(\prod_{l=i+1}^{j-1} s_l^{\hat{\mathcal{K}}}\hat{\mathcal{P}}_l\right)s_j^{\hat{\mathcal{K}}}|\bar{P}_j,\mathcal{R}_{j+1}\rangle.$$

(48)

Eq. (48) remains valid for $i = 0$ with the substitutions $\gamma^{P_0}_{\mathcal{R}_0\mathcal{R}_1} \to \Gamma^{R_1}_{A'A}$ and $\langle P_0,\bar{\mathcal{R}}_0| \to \langle A',\bar{A}|$, as well as for $j = n + 1$, with the substitutions $\gamma^{P_{n+1}}_{\mathcal{R}_{n+1}\mathcal{R}_{n+2}} \to \Gamma^{R_{n+1}}_{B'B}$ and $|\bar{P}_{n+1},\mathcal{R}_{n+2}\rangle \to |\bar{B}',B\rangle$. The matrix elements are calculated using the full set of two-Reggeon states (30), the completeness condition (31) and the definitions of the operators $\hat{\mathcal{K}}$ (43) and $\hat{\mathcal{P}}_l$ (47) and

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the matrix elements (32), (33), (34), (37), (45) and (46). After derivation of the discontinuity (27) the substitution (48) is practically evident. In the Born approximation it follows directly from the above mentioned definitions. The factors $s_i^{\hat{\mathcal{K}}}$ appear from sum of contributions of any number of intermediate particles with rapidities between y_{i-1} and y_i exactly in the same way as the factor $s^{\hat{\mathcal{K}}}$ in (44).

It completes the calculation of the discontinuities.

5 Bootstrap conditions for the Reggeon vertices

For elastic amplitudes the bootstrap relation (26) and the representation of the discontinuity (27) give

$$\langle A'\bar{A}|\, s^{\hat{\mathcal{K}}}\, |\bar{B}'B\rangle = -\delta(q_{A\perp} - q_{B\perp})2(2\pi)^{D-1}\bar{\Gamma}^{\mathcal{R}}_{A'A}\omega_{\mathcal{R}}\frac{s^{\omega_{\mathcal{R}}}}{d_{\mathcal{R}}}\Gamma^{\mathcal{R}}_{B'B}\,, \qquad (49)$$

where $\mathcal{R}$ takes values $\mathcal{G}$ and $\mathcal{Q}$. This equation is satisfied if the Reggeon vertices obey the conditions:

$$|\bar{B}'B\rangle = g|\mathcal{R}_\omega(q_{B\perp})\rangle\Gamma^{\mathcal{R}}_{B'B}\,, \quad \langle A'\bar{A}| = g\bar{\Gamma}^{\mathcal{R}}_{A'A}\langle\mathcal{R}_\omega(q_{A\perp})|\,, \qquad (50)$$

where $|\mathcal{R}_\omega(q_\perp)\rangle$ are universal (process independent) eigenstates of the kernel $\hat{\mathcal{K}}$ with the eigenvalues $\omega_{\mathcal{R}}(q)$

$$\hat{\mathcal{K}}|\mathcal{R}_\omega(q_\perp)\rangle = \omega_{\mathcal{R}}(q_\perp)|\mathcal{R}_\omega(q_\perp)\rangle\,, \quad \langle\mathcal{R}_\omega(q_\perp)|\hat{\mathcal{K}} = \langle\mathcal{R}_\omega(q_\perp)|\,\omega_{\mathcal{R}}(q_\perp)\,, \qquad (51)$$

and with scalar product

$$\langle\mathcal{R}'_\omega(q'_\perp)|\mathcal{R}_\omega(q_\perp)\rangle = -\delta_{\mathcal{R}'\mathcal{R}}\delta(q'_\perp - q_\perp)C_{\mathcal{R}}\int\frac{\mathrm{d}^{D-2}r_\perp}{d_{\mathcal{R}}(r_\perp)(q-r)^2_\perp}\,, \qquad (52)$$

where $C_{\mathcal{G}} = C_A = N_c$, $C_{\mathcal{Q}} = 2C_F = (N_c^2 - 1)/N_c$. Note that the conditions for "ket"– and "bra"–vectors in (50) and (51) are not independent, because these vectors are related with each other by the change of $+$ and $-$ momenta components.

It occurs that an infinite number of bootstrap relations for inelastic amplitudes requires besides (50)–(52) only one additional condition. This condition can be obtained from the bootstrap relation for amplitudes of the

process $A + B \to A' + P + B'$. Taking in (26) $n = 1$ and $k = 0$ and writing corresponding discontinuities according to (48), we have

$$\langle A' \bar{A}| \, s_1^{\hat{\mathcal{K}}} \left(\hat{\mathcal{P}}_1 \, s_2^{\hat{\mathcal{K}}} \, |\bar{B}'B\rangle \; + \; |\bar{P}\mathcal{R}_2\rangle \frac{s_2^{\omega_2}}{d_2} \Gamma_{B'B}^{\mathcal{R}_2} \right)$$

$$= -\delta(q_{A\perp} + k_{1\perp} + k_{2\perp} - q_{B\perp}) 2(2\pi)^{D-1} \bar{\Gamma}_{A'A}^{\mathcal{R}_1} \omega_1 \frac{s_1^{\omega_1}}{d_1} \, \gamma_{\mathcal{R}_1 \mathcal{R}_2}^{P} \, \frac{s_2^{\omega_2}}{d_2} \, \Gamma_{B'B}^{\mathcal{R}_2} . \quad (53)$$

This equality will be satisfied if together with (50)–(52) the condition

$$\hat{\mathcal{P}}_i \, |\mathcal{R}_\omega(q_{(i+1)\perp})\rangle \, g \, d_{i+1}(q_{(i+1)\perp}) + |\bar{P}_i \mathcal{R}_{i+1}\rangle = |\mathcal{R}_\omega(q_{i\perp})\rangle \, g \, \gamma_{\mathcal{R}_i \mathcal{R}_{i+1}}^{P_i} , \quad (54)$$

where $q_{i\perp} = q_{(i+1)\perp} - k_{i\perp}$, will be fulfilled. For the "bra"–vectors this condition is written as

$$g \, d_i(q_{i\perp}) \langle \mathcal{R}_\omega(q_{i\perp})|\hat{\mathcal{P}}_i \; + \; \langle P_i \mathcal{R}_i| = g \, \gamma_{\mathcal{R}_i \mathcal{R}_{i+1}}^{P_i} \langle \mathcal{R}_\omega(q_{(i+1)\perp})| , \quad (55)$$

where $q_{(i+1)\perp} = q_{i\perp} + k_{i\perp}$. Let us prove that the equalities (50)–(52) and (54), (55) secure fulfilment of all infinite set of the bootstrap relations (26). Consider the terms with $l = n$ and $l = n + 1$ in (26). Corresponding discontinuities are determined by (48). Using (50) and (51) for the $s_{k\,n+1}$–channel discontinuity we obtain that the sum of the discontinuities in the channels s_{kn} and $s_{k\,n+1}$ contains

$$g \, \hat{\mathcal{P}}_n \, |\mathcal{R}_\omega(q_{(n+1)\perp})\rangle \; + \; |\bar{P}_n \mathcal{R}_{n+1}\rangle \frac{1}{d_{n+1}} = |R_\omega(q_{n\perp})\rangle \, g \, \gamma_{\mathcal{R}_n \mathcal{R}_{n+1}}^{P_n} \frac{1}{d_{n+1}} . \quad (56)$$

Here the equation (54) was used. Now the procedure can be repeated: we can apply to this sum Eqs. (50) and (51), and to the sum of the obtained result with the $s_{k\,n-1}$–channel discontinuity Eq. (54). Thus all sum over l from $k + 1$ to $n + 1$ is reduced to one term. Quite analogous procedure (with use of the bootstrap conditions for "bra"–vectors) can be applied to the sum over l from 0 to $k - 1$. As a result we have that the left part of (26) with the coefficient $-2(2\pi)^{D-1}\delta(q_{(k+1)\perp} - q_{k\perp} - k_{k\perp})$, where $q_{(k+1)\perp} = p_{B\perp} - p_{B'\perp} - \sum_{l=k+1}^{n} k_{l\perp}$, $q_{k\perp} = p_{A'\perp} - p_{A\perp} + \sum_{l=1}^{k-1} k_{l\perp}$, can be obtained from the r.h.s. of (6) by the replacement

$$\gamma_{\mathcal{R}_k \mathcal{R}_{k+1}}^{P_k} \longrightarrow \langle P_k \mathcal{R}_k|\mathcal{R}_\omega(q_{(k+1)\perp})\rangle g d_{k+1} - g d_k \langle \mathcal{R}_\omega(q_{k\perp})|\bar{P}_k \mathcal{R}_{k+1}\rangle . \quad (57)$$

Taking difference of (54) multiplied by $g \, d_i \langle \mathcal{R}_\omega(q_{i\perp})|$ and (55) multiplied by $|\mathcal{R}_\omega(q_{(i+1)\perp})\rangle g \, d_{i+1}$ and using the normalization (52) we obtain

$$\langle P_k \mathcal{R}_k|\mathcal{R}_\omega(q_{k+1})\rangle \, g \, d_{k+1} - g \, d_k \, \langle \mathcal{R}_\omega(q_k)|\bar{P}_k \mathcal{R}_{k+1}\rangle$$

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$$= -2(2\pi)^{D-1}\delta(q_{(k+1)\perp} - q_{k\perp} - k_{k\perp})$$

$$\times \left(\gamma^{P_k}_{\mathcal{R}_k \mathcal{R}_{k+1}} \omega_{\mathcal{R}_{k+1}}(q_{k+1}) - \omega_{\mathcal{R}_k}(q_k) \gamma^{P_k}_{\mathcal{R}_k \mathcal{R}_{k+1}} \right). \qquad (58)$$

It concludes the proof.

Thus, fulfilment of the bootstrap conditions (50)–(51) and (54), (55) secures implementation of all infinite set of the bootstrap relations (26).

6   Verification of bootstrap conditions on Reggeon vertices

Let us start from the impact factors. As it was already mentioned, the conditions for "ket"– and "bra"–vectors are not independent, so that in the following we consider only "ket"–vectors. Using the PPR vertices (7) and the definition of the vertices $\underline{\Gamma}$ given after (32) it is easy to obtain

$$\frac{1}{2p_G^-}\sum_P \left(\Gamma^{\mathcal{G}_2}_{G'P}\Gamma^{\mathcal{G}_1}_{PG} - \underline{\Gamma}^{\mathcal{G}_2}_{GP}\underline{\Gamma}^{\mathcal{G}_1}_{PG'} \right) = -2g^2 T^{\mathcal{G}}_{\mathcal{G}_1\mathcal{G}_2}\, p_G^- T^{\mathcal{G}}_{G'G}(e^*_{G'_\perp}e_{G\perp})\,, \quad (59)$$

$$\frac{1}{2p_Q^-}\sum_P \left(\Gamma^{\mathcal{G}_2}_{Q'P}\Gamma^{\mathcal{G}_1}_{PQ} - \underline{\Gamma}^{\mathcal{G}_2}_{QP}\underline{\Gamma}^{\mathcal{G}_1}_{PQ'} \right) = g^2 T^{\mathcal{G}}_{\mathcal{G}_1\mathcal{G}_2}\, \bar{u}_{Q'}t^{\mathcal{G}}\gamma^- u_Q\,, \qquad (60)$$

$$\frac{1}{2p_{\bar{Q}}^-}\sum_P \left(\Gamma^{\mathcal{G}_2}_{\bar{Q}'P}\Gamma^{\mathcal{G}_1}_{P\bar{Q}} - \underline{\Gamma}^{\mathcal{G}_2}_{\bar{Q}P}\underline{\Gamma}^{\mathcal{G}_1}_{PQ'} \right) = -g^2 T^{\mathcal{G}}_{\mathcal{G}_1\mathcal{G}_2}\, \bar{v}_{\bar{Q}}t^{\mathcal{G}}\gamma^- v_{\bar{Q}'}\,. \qquad (61)$$

Clearly, in the first of these equations intermediate particles P are gluons, in the first (second) term of the second equation they are quarks (antiquarks) and in the third equation vise versa. Note that the important fact of disappearance of all t–channel colour states besides the colour octet one is provided by the signaturization. All these three equations can be presented as

$$\frac{1}{2p_B^-}\sum_P \left(\Gamma^{\mathcal{G}}_{B'P}\Gamma^{\mathcal{G}_1}_{PB} - \underline{\Gamma}^{\mathcal{G}_2}_{\bar{B}P}\underline{\Gamma}^{\mathcal{G}_1}_{P\bar{B}'} \right) = g T^{\mathcal{G}}_{\mathcal{G}_1\mathcal{G}_2}\, \Gamma^{\mathcal{G}_2}_{B'B}\,. \qquad (62)$$

Consequently, according to the definition (32), for the case of boson-type t–channel states the bootstrap condition (50) is fulfilled, and the universal state $|\mathcal{R}_\omega(q_\perp)\rangle$, which we call in this case $|\mathcal{G}_\omega(q_\perp)\rangle$, is defined by the matrix elements

$$\langle \mathcal{G}_1 \mathcal{G}_2 | \mathcal{G}_\omega(q_\perp) \rangle = \delta(r_{1\perp} + r_{2\perp} - q_\perp) T^{\mathcal{G}}_{\mathcal{G}_1\mathcal{G}_2}. \qquad (63)$$

In the following we will show that this state is the eigenstate of the kernel $\hat{\mathcal{K}}$ with the eigenvalues $\omega_{\mathcal{G}}(q)$. Now we turn to the fermion-type t–channel states.

Using the PPR vertices (8) and (7) we obtain

$$\frac{1}{2p_G^-}\sum_P\left(\Gamma^{\mathcal{Q}_2}_{\bar{Q}'P}\Gamma^{\mathcal{G}_1}_{PG} \; + \; \underline{\Gamma}^{\mathcal{Q}_2}_{\bar{G}P}\underline{\Gamma}^{\mathcal{G}_1}_{PQ'}\right) = -g^2 t^{\mathcal{G}_1}t^G \hat{e}_{G\perp}v_{\bar{Q}'} = gt^{\mathcal{G}_1}\Gamma^{\mathcal{Q}_2}_{\bar{Q}'G}. \quad (64)$$

Evidently, here in the first term intermediate particles are gluons and in the second – quarks. Note that due to the signaturization only t–channel colour triplet does survive. To obtain (64) one needs to perform commutations of gamma matrices and to omit leftmost matrices γ^-, that can be done due to the strong ordering (1). The same we will do in the following with rightmost γ^+.

In the same way we obtain

$$\frac{1}{2p_Q^-}\sum_P\left(\Gamma^{\mathcal{Q}_2}_{G'P}\Gamma^{\mathcal{G}_1}_{PQ} \; + \; \underline{\Gamma}^{\mathcal{Q}_2}_{\bar{Q}P}\underline{\Gamma}^{\mathcal{G}_1}_{PG'}\right) = -g^2 t^{\mathcal{G}_1}t^{G'} \hat{e}^*_{G'\perp}u_Q = gt^{\mathcal{G}_1}\Gamma^{\mathcal{Q}_1}_{G'Q}, \quad (65)$$

and

$$\frac{1}{2p_G^-}\sum_P\left(\Gamma^{\mathcal{G}_2}_{\bar{Q}'P}\Gamma^{\mathcal{Q}_1}_{PG} \; + \; \underline{\Gamma}^{\mathcal{G}_2}_{\bar{G}P}\underline{\Gamma}^{\mathcal{Q}_1}_{PQ'}\right) = g^2 t^{\mathcal{G}_2}t^G \hat{e}_{G\perp}v_{\bar{Q}'} = -gt^{\mathcal{G}_2}\Gamma^{\mathcal{Q}_2}_{\bar{Q}'G}. \quad (66)$$

$$\frac{1}{2p_Q^-}\sum_P\left(\Gamma^{\mathcal{G}_2}_{G'P}\Gamma^{\mathcal{Q}_1}_{PQ} \; + \; \underline{\Gamma}^{\mathcal{G}_2}_{\bar{Q}P}\underline{\Gamma}^{\mathcal{Q}_1}_{PG'}\right) = g^2 t^{\mathcal{G}_2}t^{G'} \hat{e}^*_{G'\perp}u_Q = -gt^{\mathcal{G}_2}\Gamma^{\mathcal{Q}_1}_{G'Q}. \quad (67)$$

These equations give

$$\frac{1}{2p_B^-}\sum_P\left(\Gamma^{\mathcal{Q}_2}_{B'P}\Gamma^{\mathcal{G}_1}_{PB} \; + \; \underline{\Gamma}^{\mathcal{Q}_2}_{\bar{B}P}\underline{\Gamma}^{\mathcal{G}_1}_{PB'}\right) = gt^{\mathcal{G}_1}\Gamma^{\mathcal{Q}_1}_{B'B}, \quad (68)$$

$$\frac{1}{2p_B^-}\sum_P\left(\Gamma^{\mathcal{G}_2}_{B'P}\Gamma^{\mathcal{Q}_1}_{PB} \; + \; \underline{\Gamma}^{\mathcal{G}_2}_{\bar{B}P}\underline{\Gamma}^{\mathcal{Q}_1}_{PB'}\right) = -gt^{\mathcal{G}_2}\Gamma^{\mathcal{Q}_1}_{B'B}. \quad (69)$$

According to the definition (32), the bootstrap condition (50) is fulfilled for the case of fermion-type t–channel states also, with the universal state $|\mathcal{Q}_\omega(q_\perp)\rangle$, defined by its matrix elements

$$\langle\mathcal{G}_1\mathcal{Q}_2|\mathcal{Q}_\omega(q_\perp)\rangle = \delta(r_{1\perp}+r_{2\perp}-q_\perp)t^{\mathcal{G}_1}, \quad (70)$$

$$\langle\mathcal{Q}_1\mathcal{G}_2|\mathcal{Q}_\omega(q_\perp)\rangle = -\delta(r_{1\perp}+r_{2\perp}-q_\perp)t^{\mathcal{G}_2}. \quad (71)$$

Now let us demonstrate that the states $|\mathcal{G}_\omega(q_\perp)\rangle$ and $|\mathcal{Q}_\omega(q_\perp)\rangle$ are the eigenstate of the kernel $\hat{\mathcal{K}}$ with the eigenvalues $\omega_{\mathcal{G}}(q_\perp)$ and $\omega_{\mathcal{Q}}(q_\perp)$ correspondingly. First we need to obtain explicit expressions for matrix elements of the operator $\hat{\mathcal{K}}_r$ (34). For matrix elements between states of two Reggeized gluons we obtain, using the vertices (11) (actually it is much more convenient to take them in any of the gauges (14), (15)) and the definition of $\gamma_G^{\mathcal{G}_1\mathcal{G}_2}$ given just after (34):

$$\langle \mathcal{G}_1\mathcal{G}_2 | \hat{\mathcal{K}}_r | \mathcal{G}_1'\mathcal{G}_2'\rangle = \delta(r_{1\perp} + r_{2\perp} - r_{1\perp}' - r_{2\perp}') \frac{1}{2(2\pi)^{D-1}} \sum_G \gamma_{\mathcal{G}_1\mathcal{G}_1'}^G \gamma_G^{\mathcal{G}_2\mathcal{G}_2'}$$

$$= \delta(r_{1\perp} + r_{2\perp} - r_{1\perp}' - r_{2\perp}') \mathrm{K}_r^{GG}(r_{1\perp}, r_{2\perp}; r_{1\perp}', r_{2\perp}') \frac{2 T_{\mathcal{G}_1\mathcal{G}_1'}^a T_{\mathcal{G}_2'\mathcal{G}_2}^a}{N_c}, \quad (72)$$

$$\mathrm{K}_r^{GG}(r_1, r_2; r_{1\perp}', r_{2\perp}') = \frac{g^2}{(2\pi)^{D-1}} \frac{N_c}{2} \left((r_{1\perp} + r_{2\perp})^2 - \frac{r_{1\perp}^2 r_{2\perp}'^2 + r_{2\perp}^2 r_{1\perp}'^2}{(r_1 - r_1')_\perp^2} \right). \tag{73}$$

Now it is easy to see that the state $|\mathcal{G}_\omega(q_\perp)\rangle$ is the eigenstate of $\hat{\mathcal{K}}$ (43). Indeed, since this operator conserves fermion number, it is sufficient, with account of the completeness condition (31) and (63), to show that

$$\langle \mathcal{G}_1\mathcal{G}_2 | \hat{\mathcal{K}}_r | \mathcal{G}_1'\mathcal{G}_2'\rangle \langle \mathcal{G}_1'\mathcal{G}_2' | \mathcal{G}_\omega(q_\perp)\rangle$$

$$= \delta(r_{1\perp} + r_{2\perp} - q) \left(\omega_{\mathcal{G}}(q_\perp) - \omega_{\mathcal{G}}(r_{2\perp}) - \omega_{\mathcal{G}}(r_{1\perp}) \right) T_{\mathcal{G}_1\mathcal{G}_2}^{\mathcal{G}}. \tag{74}$$

Using (31), (72) and (73) it is easy to obtain

$$\langle \mathcal{G}_1\mathcal{G}_2 | \hat{\mathcal{K}}_r | \mathcal{G}_1'\mathcal{G}_2'\rangle \langle \mathcal{G}_1'\mathcal{G}_2' | \mathcal{G}_\omega(q_\perp)\rangle = \delta(r_{1\perp} + r_{2\perp} - q) \frac{N_c}{2} \frac{g^2}{(2\pi)^{D-1}}$$

$$\times \int \frac{\mathrm{d}^{D-2}k_\perp}{k_\perp^2} \left(\frac{q_\perp^2}{(q-k)_\perp^2} - \frac{r_{1\perp}^2}{(r_{1\perp} - k)_\perp^2} - \frac{r_{2\perp}^2}{(r_{2\perp} - k)_\perp^2} \right) T_{\mathcal{G}_1\mathcal{G}_2}^{\mathcal{G}}. \tag{75}$$

Using the representation (18) for trajectories in (77), we see that it is satisfied, i.e. indeed $|\mathcal{G}_\omega(q_\perp)\rangle$ is the eigenstate of the kernel with the eigenvalue $\omega_{\mathcal{G}}(q_\perp)$.

Turn now to the fermion-type states. The matrix elements between the states of Reggeized gluon and Reggeized quark correspond to

$$\langle \mathcal{Q}_1\mathcal{G}_2 | \hat{\mathcal{K}}_r | \mathcal{Q}_1'\mathcal{G}_2'\rangle$$

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$$= \langle \mathcal{G}_2 \mathcal{Q}_1 | \hat{\mathcal{K}}_r | \mathcal{G}_2' \mathcal{Q}_1' \rangle = \delta(r_{1\perp} + r_{2\perp} - r_{1\perp}' - r_{2\perp}') \frac{1}{2(2\pi)^{D-1}} \sum_G \gamma_{\mathcal{Q}_1 \mathcal{Q}_1'}^G \gamma_G^{\mathcal{G}_1 \mathcal{G}_1'}$$

$$= \delta(r_{1\perp} + r_{2\perp} - r_{1\perp}' - r_{2\perp}') \mathrm{K}_r^{QG}(r_{1\perp}, r_{2\perp}; r_{1\perp}', r_{2\perp}') \frac{2 T_{\mathcal{G}_2' \mathcal{G}_2}^a t^a}{N_c} , \qquad (76)$$

where

$$\mathrm{K}_r^{QG}(r_{1\perp}, r_{2\perp}; r_{1\perp}', r_{2\perp}')$$

$$= \frac{g^2}{(2\pi)^{D-1}} \frac{N_c}{2} \left(m - \hat{r}_{1\perp} - \hat{r}_{2\perp} - \frac{(m - \hat{r}_{1\perp}) r_{2\perp}'^2 + (m - \hat{r}_{1\perp}') r_{2\perp}^2}{(r_1 - r_1')_\perp^2} \right) , \quad (77)$$

and

$$\langle \mathcal{G}_1 \mathcal{Q}_2 | \hat{\mathcal{K}}_r | \mathcal{Q}_1' \mathcal{G}_2' \rangle$$

$$= \langle \mathcal{Q}_2 \mathcal{G}_1 | \hat{\mathcal{K}}_r | \mathcal{G}_2' \mathcal{Q}_1' \rangle = \delta(r_{1\perp} + r_{2\perp} - r_{1\perp}' - r_{2\perp}') \frac{1}{2(2\pi)^{D-1}} \sum_Q \gamma_{\mathcal{G}_1 \mathcal{Q}_1'}^Q \gamma_Q^{\mathcal{Q}_2 \mathcal{G}_2'}$$

$$= \delta(r_{1\perp} + r_{2\perp} - r_{1\perp}' - r_{2\perp}') \mathrm{K}_r^{GQ}(r_{1\perp}, r_{2\perp}; r_{1\perp}', r_{2\perp}') (2N_c) t^{\mathcal{G}_2'} t^{\mathcal{G}_1} , \qquad (78)$$

where

$$\mathrm{K}_r^{GQ}(r_{1\perp}, r_{2\perp}; r_{1\perp}', r_{2\perp}')$$

$$= \frac{g^2}{(2\pi)^{D-1}} \frac{-1}{2N_c} \left(m - \hat{r}_{1\perp} - \hat{r}_{2\perp} - (m - \hat{r}_{2\perp}) \frac{1}{m - (\hat{r}_{1\perp}' - \hat{r}_{1\perp})} (m - \hat{r}_{1\perp}') \right) . \qquad (79)$$

In order to prove that the state $|\mathcal{Q}_\omega(q_\perp)\rangle$ is the eigenstate of $\hat{\mathcal{K}}$ (43) we need, taking into account (70) and (71), to show that

$$\langle \mathcal{G}_1 \mathcal{Q}_2 | \hat{\mathcal{K}}_r | \mathcal{Q}_\omega(q_\perp) \rangle = -\langle \mathcal{Q}_2 \mathcal{G}_1 | \hat{\mathcal{K}}_r | \mathcal{Q}_\omega(q_\perp) \rangle$$

$$= \delta(r_{1\perp} + r_{2\perp} - q) \left(\omega_\mathcal{Q}(q_\perp) - \omega_\mathcal{G}(r_{1\perp}) - \omega_\mathcal{Q}(r_{2\perp}) \right) t^{\mathcal{G}_1}. \qquad (80)$$

Using (31), (76)–(79), and (70), (71), it is easy to obtain

$$\langle \mathcal{G}_1 \mathcal{Q}_2 | \hat{\mathcal{K}}_r | \mathcal{Q}_\omega(q_\perp) \rangle$$

$$= \langle \mathcal{G}_1 \mathcal{Q}_2 | \hat{\mathcal{K}}_r | \mathcal{G}_1' \mathcal{Q}_2' \rangle \langle \mathcal{G}_1' \mathcal{Q}_2' | \mathcal{Q}_\omega(q_\perp) \rangle + \langle \mathcal{G}_1 \mathcal{Q}_2 | \hat{\mathcal{K}}_r | \mathcal{Q}_1' \mathcal{G}_2' \rangle \langle \mathcal{Q}_1' \mathcal{G}_2' | \mathcal{Q}_\omega(q_\perp) \rangle$$

$$= \delta(r_{1\perp} + r_{2\perp} - q_\perp) g^2$$

$$\int \frac{\mathrm{d}^{D-2}k_\perp}{2(2\pi)^{D-1}} \left(\left(\frac{(m-\hat{q}_\perp)}{(q-k)_\perp^2} - \frac{(m-\hat{r}_{2\perp})}{(r_2-k)_\perp^2} \right) \frac{N_c^2-1}{N_c(m-\hat{k}_\perp)} - \frac{N_c r_{1\perp}^2}{k_\perp^2(r_1-k)_\perp^2} \right) t^{\mathcal{G}_1}.$$

(81)

Together with the representation (18) it shows that (80) is satisfied, i.e. $|\mathcal{Q}_\omega(q_\perp)\rangle$ is the eigenstate of the kernel with the eigenvalue $\omega_{\mathcal{Q}}(q_\perp)$.

The normalization conditions (52) follow immediately from (63), (70), (71).

Thus, we have demonstrated that the conditions (50)–(52) are satisfied. Let us turn now to the last conditions. We will consider the bootstrap conditions for "ket"–vectors (54) and use the light-cone gauge (10). First we need to find explicit expressions for the impact factors of Reggeon-particle transitions and the matrix elements of the production operator $\hat{\mathcal{P}}_l$ between the vectors $|\mathcal{R}_\omega(q_\perp)\rangle$ and two-Reggeon states. The expressions for the impact factors are calculated using their definition (45) and the vertices (7), (8), (15)–(17). To find the matrix elements of the production operator we use the definitions of $\hat{\mathcal{P}}_l$ (47), the vectors (63), (70), (71) and the completeness condition (31).

Let us start with gluon production. In the case of boson-type q_{j+1}-channel we obtain for the impact factor

$$\langle \mathcal{G}_1 \mathcal{G}_2 | \bar{G}_j \mathcal{G}_{j+1} \rangle = \delta(r_{1\perp} + r_{2\perp} - q_{j\perp}) \frac{1}{2k_j^-} \sum_G \left(\Gamma_{\bar{G}_j G}^{\mathcal{G}_2} \gamma_{\mathcal{G}_1 \mathcal{G}_{j+1}}^G + \mathbb{\Gamma}_{GG_j}^{\mathcal{G}_1} \gamma_G^{\mathcal{G}_2 \mathcal{G}_{j+1}} \right)$$

$$= \delta(r_{1\perp} + r_{2\perp} - q_{j\perp}) 2g^2 e_{G_j\perp}^*$$

$$\times \left(- \left(T^{\mathcal{G}_{j+1}} T^{G_j} \right)_{\mathcal{G}_1 \mathcal{G}_2} \left(q_{(j+1)\perp} - (q_{(j+1)} - r_1)_\perp \frac{q_{(j+1)\perp}^2}{(q_{(j+1)} - r_1)_\perp^2} \right) \right.$$

$$\left. + \left(T^{G_j} T^{\mathcal{G}_{j+1}} \right)_{\mathcal{G}_1 \mathcal{G}_2} \left(q_{(j+1)\perp} - (q_{(j+1)} - r_2)_\perp \frac{q_{(j+1)\perp}^2}{(q_{(j+1)} - r_2)_\perp^2} \right) \right). \quad (82)$$

For corresponding martix element of the kernel only two-gluon intermediate states in the completeness condition contribute, with the result

$$\langle \mathcal{G}_1 \mathcal{G}_2 | \hat{\mathcal{G}}_j | \mathcal{G}_\omega(q_{(j+1)\perp}) \rangle = \delta(q_{(j+1)\perp} - k_{j\perp} - q_{j\perp}) 2g e_{G_j\perp}^*$$

$$\times \left(\left(-T^{\mathcal{G}_{j+1}} T^{G_j} \right)_{\mathcal{G}_1 \mathcal{G}_2} \left(\frac{(q_{(j+1)} - r_1)_\perp}{(q_{(j+1)} - r_1)_\perp^2} - \frac{k_{j\perp}}{k_{j\perp}^2} \right) \right.$$

$$+ \left(T^{G_j} T^{G_{j+1}} \right)_{\mathcal{G}_1 \mathcal{G}_2} \left(\frac{(q_{(j+1)} - r_2)_\perp}{(q_{(j+1)} - r_2)^2_\perp} - \frac{k_{j\perp}}{k^2_{j\perp}} \right) \right) . \tag{83}$$

Now, with account of (63), it is quite easy to obtain

$$\langle \mathcal{G}_1 \mathcal{G}_2 | \hat{\mathcal{G}}_i | \mathcal{G}_\omega (q_{(i+1)\perp}) \rangle \, g \, d_{i+1} (q_{(i+1)\perp}) + \langle \mathcal{G}_1 \mathcal{G}_2 | \bar{G}_i \mathcal{G}_{i+1} \rangle$$

$$= \langle \mathcal{G}_1 \mathcal{G}_2 | \mathcal{G}_\omega (q_{i\perp}) \rangle \, g \, \gamma^{G_i}_{\mathcal{G}_i \mathcal{G}_{i+1}} , \tag{84}$$

which proves the bootstrap condition (54) for this case.

Another possibility for gluon production is fermion–type q_{j+1}-channel. In this case we have to consider projections on two different two-Reggeon states: $\langle \mathcal{Q}_1 \mathcal{G}_2 |$ and $\langle \mathcal{G}_1 \mathcal{Q}_2 |$. For the first one we obtain

$$\langle \mathcal{Q}_1 \mathcal{G}_2 | \bar{G}_j \mathcal{Q}_{j+1} \rangle$$

$$= \delta(r_{1\perp} + r_{2\perp} - q_{j\perp}) \frac{1}{2k^-_j} \left(\sum_G \Gamma^{\mathcal{G}_2}_{G_j G} \gamma^G_{\mathcal{Q}_1 \mathcal{Q}_{j+1}} + \sum_{\bar{Q}} \underline{\Gamma}^{\mathcal{Q}_1}_{\bar{Q} G_j} \gamma^{\mathcal{G}_2 \mathcal{Q}_{j+1}}_{\bar{Q}} \right)$$

$$= \delta(r_{1\perp} + r_{2\perp} - q_{j\perp}) g^2 e^*_{G_j \perp} \left([t^{\mathcal{G}_2} t^{G_j}] \left(\gamma_\perp + 2(m - \hat{q}_{(j+1)\perp}) \frac{(k_j + r_2)_\perp}{(k_j + r_2)^2_\perp} \right) \right.$$

$$\left. + t^{G_j} t^{\mathcal{G}_2} \gamma_\perp \frac{1}{m - (\hat{k}_j + \hat{r}_1)_\perp} \hat{r}_{2\perp} \right) . \tag{85}$$

Calculating corresponding matrix element of the kernel one needs to take again only intermediate states of one type ($|\mathcal{Q}\mathcal{G}\rangle$) in the completeness condition. The result is

$$\langle \mathcal{Q}_1 \mathcal{G}_2 | \hat{\mathcal{G}}_j | \mathcal{Q}_\omega (q_{(j+1)\perp}) \rangle$$

$$= \delta(q_{(j+1)\perp} - k_{j\perp} - q_{j\perp}) g e^*_{G_j \perp} \left(2[t^{G_j} t^{\mathcal{G}_2}] \left(\frac{(k_j + r_2)_\perp}{(k_j + r_2)^2_\perp} - \frac{k_{j\perp}}{k^2_{j\perp}} \right) \right.$$

$$\left. + t^{G_j} t^{\mathcal{G}_2} \left(\gamma_\perp + 2(m - (\hat{k}_j + \hat{r}_1)_\perp) \frac{k_{j\perp}}{k^2_{j\perp}} \right) \frac{1}{(m - (\hat{k}_j + \hat{r}_1)_\perp)} \right) , \tag{86}$$

so that, with account of (71), we obtain

$$\langle \mathcal{Q}_1 \mathcal{G}_2 | \hat{\mathcal{G}}_i | \mathcal{Q}_\omega (q_{(i+1)\perp}) \rangle \, g \, d_{i+1} (q_{(i+1)\perp}) + \langle \mathcal{Q}_1 \mathcal{G}_2 | \bar{G}_i \mathcal{Q}_{i+1} \rangle$$

$$= \langle \mathcal{Q}_1 \mathcal{G}_2 | \mathcal{Q}_\omega (q_{i\perp}) \rangle \, g \, \gamma^{G_i}_{\mathcal{Q}_i \mathcal{Q}_{i+1}} , \tag{87}$$

so that the bootstrap condition (54) is also fulfilled for this case.

The projection on the state $\langle \mathcal{G}_1 \mathcal{Q}_2|$ is considered quite analogously. We obtain

$$\langle \mathcal{G}_1 \mathcal{Q}_2 | \bar{G}_j \mathcal{Q}_{j+1} \rangle$$

$$= \delta(r_{1\perp} + r_{2\perp} - q_{j\perp}) \frac{1}{2k_j^-} \left(\sum_Q \Gamma_{\bar{G}_j Q}^{\mathcal{Q}_2} \gamma_{\mathcal{G}_1 \mathcal{Q}_{j+1}}^Q + \sum_G \Gamma_{\bar{G}G_j}^{\mathcal{G}_1} \gamma_G^{\mathcal{Q}_2 \mathcal{Q}_{j+1}} \right)$$

$$= \delta(r_{1\perp} + r_{2\perp} - q_{j\perp}) g^2 e_{G_j\perp}^* \left([t^{G_j} t^{\mathcal{G}_1}] \left(\gamma_\perp + 2(m - \hat{q}_{(j+1)\perp}) \frac{(k_j + r_1)_\perp}{(k_j + r_1)_\perp^2} \right) \right.$$

$$\left. -t^{G_j} t^{\mathcal{G}_1} \gamma_\perp \frac{1}{m - (\hat{k}_j + \hat{r}_2)_\perp} \hat{r}_{1\perp} \right) , \qquad (88)$$

and

$$\langle \mathcal{G}_1 \mathcal{Q}_2 | \hat{\mathcal{G}}_j | \mathcal{Q}_\omega(q_{(j+1)\perp}) \rangle$$

$$= \delta(q_{(j+1)\perp} - k_{j\perp} - q_{j\perp}) g e_{G_j\perp}^* \left(2[t^{\mathcal{G}_1} t^{G_j}] \left(\frac{(k_j + r_1)_\perp}{(k_j + r_1)_\perp^2} - \frac{k_{j\perp}}{k_{j\perp}^2} \right) \right.$$

$$\left. -t^{G_j} t^{\mathcal{G}_1} \left(\gamma_\perp + 2(m - (\hat{k}_j + \hat{r}_2)_\perp) \frac{k_{j\perp}}{k_{j\perp}^2} \right) \frac{1}{(m - (\hat{k}_j + \hat{r}_2)_\perp)} \right) , \qquad (89)$$

so that, with account of (70),

$$\langle \mathcal{G}_1 \mathcal{Q}_2 | \hat{\mathcal{G}}_i | \mathcal{Q}_\omega(q_{(i+1)\perp}) \rangle \, g \, d_{i+1}(q_{(i+1)\perp}) + \langle \mathcal{G}_1 \mathcal{Q}_2 | \bar{G}_i \mathcal{Q}_{i+1} \rangle$$

$$= \langle \mathcal{G}_1 \mathcal{Q}_2 | \mathcal{Q}_\omega(q_{i\perp}) \rangle \, g \, \gamma_{\mathcal{Q}_i \mathcal{Q}_{i+1}}^{G_i} , \qquad (90)$$

and we see that the bootstrap condition is also satisfied.

Let us consider now antiquark production. Here again we have to consider projections on the states $\langle \mathcal{Q}_1 \mathcal{G}_2|$ and $\langle \mathcal{G}_1 \mathcal{Q}_2|$. In the first case we obtain for the impact factor

$$\langle \mathcal{Q}_1 \mathcal{G}_2 | Q_j \mathcal{G}_{j+1} \rangle$$

$$= \delta(r_{1\perp} + r_{2\perp} - q_{j\perp}) \frac{1}{2k_j^-} \left(\sum_{\bar{Q}} \Gamma_{Q_j \bar{Q}}^{\mathcal{G}_2} \gamma_{\mathcal{Q}_1 G_{j+1}}^{\bar{Q}} - \sum_G \Gamma_{\bar{G}Q_j}^{\mathcal{Q}_1} \gamma_G^{\mathcal{G}_2 \mathcal{G}_{j+1}} \right)$$

$$= \delta(r_{1\perp} + r_{2\perp} - q_{j\perp}) \frac{g^2}{k_j^-}$$

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$$\times \left(t^{\mathcal{G}_{j+1}} t^{\mathcal{G}_2} \hat{q}_{(j+1)\perp} - [t^{\mathcal{G}_{j+1}} t^{\mathcal{G}_2}] \left(\hat{q}_{(j+1)\perp} - (\hat{k}_j + \hat{r}_1)_\perp \frac{q^2_{(j+1)\perp}}{(k_j + r_1)^2_\perp} \right) \right) v_{\bar{\mathcal{Q}}_j} \,.$$

(91)

In the matrix element of the kernel now one needs to take intermediate states of the type $|\mathcal{GG}\rangle$. We obtain

$$\langle \mathcal{Q}_1 \mathcal{G}_2 | \hat{\bar{\mathcal{Q}}}_j | G_\omega(q_{(j+1)\perp}) \rangle = \delta(q_{(j+1)\perp} - k_{j\perp} - q_{j\perp}) \frac{g}{k_j^-} [t^{\mathcal{G}_2} t^{\mathcal{G}_{j+1}}] \frac{\hat{r}_{1\perp} + \hat{k}_{j\perp}}{(r_1 + k_j)^2_\perp} v_{\bar{\mathcal{Q}}_j} \,,$$

(92)

so that, with account of (71), we see that the bootstrap condition (54) for this case is also fulfilled,

$$\langle \mathcal{Q}_1 \mathcal{G}_2 | \hat{\bar{\mathcal{Q}}}_j | \mathcal{G}_\omega(q_{(j+1)\perp}) \rangle g\, d_{i+1}(q_{(i+1)\perp}) + \langle \mathcal{Q}_1 \mathcal{G}_2 | Q_j \mathcal{G}_{j+1} \rangle$$

$$= \langle \mathcal{Q}_1 \mathcal{G}_2 | \mathcal{Q}_\omega(q_{i\perp}) \rangle \, g\, \gamma^{\bar{Q}_i}_{\mathcal{Q}_i \mathcal{G}_{i+1}} \,,$$

(93)

which proves the bootstrap condition (54) for this case.

The projection on the $\langle \mathcal{G}\mathcal{Q}|$–state is considered quite similarly. We have

$$\langle \mathcal{G}_1 \mathcal{Q}_2 | Q_j \mathcal{G}_{j+1} \rangle$$

$$= \delta(r_{1\perp} + r_{2\perp} - q_{j\perp}) \frac{1}{2k_j^-} \left(\sum_G \Gamma^{\mathcal{Q}_2}_{\bar{Q}_j G} \gamma^G_{\mathcal{G}_1 \mathcal{G}_{j+1}} - \sum_Q \underline{\Gamma}^{\mathcal{G}_1}_{Q Q_j} \gamma^{\mathcal{Q}_2 \mathcal{G}_{j+1}}_Q \right)$$

$$= \delta(r_{1\perp} + r_{2\perp} - q_{j\perp})$$

$$\times \frac{g^2}{k_j^-} \left([t^{\mathcal{G}_{j+1}} t^{\mathcal{G}_1}] \left(\hat{q}_{(j+1)\perp} - (\hat{k}_j + \hat{r}_2)_\perp \frac{q^2_{(j+1)\perp}}{(k_j + r_2)^2_\perp} \right) - t^{\mathcal{G}_{j+1}} t^{\mathcal{G}_1} \hat{q}_{(j+1)\perp} \right) v_{\bar{\mathcal{Q}}_j} \,,$$

(94)

$$\langle \mathcal{G}_1 \mathcal{Q}_2 | \hat{\bar{\mathcal{Q}}}_j | G_\omega(q_{(j+1)\perp}) \rangle = \delta(q_{(j+1)\perp} - k_{j\perp} - q_{j\perp}) \frac{g}{k_j^-} [t^{\mathcal{G}_{j+1}} t^{\mathcal{G}_1}] \frac{\hat{r}_{2\perp} + \hat{k}_{j\perp}}{(r_2 + k_j)^2_\perp} v_{\bar{\mathcal{Q}}_j} \,,$$

(95)

and therefore

$$\langle \mathcal{G}_1 \mathcal{Q}_2 | \hat{\bar{\mathcal{Q}}}_j | \mathcal{G}_\omega(q_{(j+1)\perp}) \rangle g\, d_{i+1}(q_{(i+1)\perp}) + \langle \mathcal{G}_1 \mathcal{Q}_2 | Q_j \mathcal{G}_{j+1} \rangle$$

$$= \langle \mathcal{G}_1 \mathcal{Q}_2 | \mathcal{Q}_\omega(q_{i\perp}) \rangle \, g\, \gamma^{\bar{Q}_i}_{\mathcal{Q}_i \mathcal{G}_{i+1}} \,.$$

(96)

Finally, we consider quark production. Here we need to consider only progection on $\langle \mathcal{G}_1 \mathcal{G}_2 |$–state. It is easy to obtain

$$\langle \mathcal{G}_1 \mathcal{G}_2 | \bar{Q}_j \mathcal{Q}_{j+1} \rangle$$

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$$= \delta(r_{1\perp} + r_{2\perp} - q_{j\perp}) \frac{1}{2k_j^-} \left(\sum_Q \Gamma^{\mathcal{G}_2}_{\mathcal{Q}_j Q} \gamma^Q_{\mathcal{G}_1 \mathcal{Q}_{j+1}} - (-1) \sum_{\bar{Q}} \underline{\Gamma}^{\mathcal{G}_1}_{\bar{Q} \bar{Q}_j} \gamma^{\mathcal{G}_2 \mathcal{Q}_{j+1}}_{\bar{Q}} \right)$$

$$= \delta(r_{1\perp} + r_{2\perp} - q_{j\perp}) \frac{g^2}{k_j^+} \bar{u}_{Q_j} \left(t^{\mathcal{G}_2} t^{\mathcal{G}_1} \hat{r}_{1\perp} - t^{\mathcal{G}_1} t^{\mathcal{G}_2} \hat{r}_{2\perp} \right) , \qquad (97)$$

where the additional (-1) is introduced to avoid double counting caused by the fact that the vertices $\underline{\Gamma}^{\mathcal{G}_1}_{\bar{Q} \bar{Q}_j}$ and $\gamma^{\mathcal{G}_2 \mathcal{Q}_{j+1}}_{\bar{Q}}$ are both include -1 for antiquark in the intermediate state. Here in the matrix element of the kernel intermediate states $|\mathcal{QG}\rangle$ and $|\mathcal{GQ}\rangle$ contribute. The result is

$$\langle \mathcal{G}_1 \mathcal{G}_2 | \hat{\mathcal{Q}}_j | \mathcal{Q}_\omega(q_{(j+1)\perp}) \rangle = \delta(q_{(j+1)\perp} - k_{j\perp} - q_{j\perp}) \frac{g}{k_j^+} \bar{u}_Q$$

$$\times \left(t^{\mathcal{G}_2} t^{\mathcal{G}_1} \hat{r}_{2\perp} \frac{1}{m - (\hat{k}_{j\perp} + \hat{r}_{2\perp})} - t^{\mathcal{G}_1} t^{\mathcal{G}_2} \hat{r}_{1\perp} \frac{1}{m - (\hat{k}_{j\perp} + \hat{r}_{1\perp})} \right) . \qquad (98)$$

With account of (63) we obtain,

$$\langle \mathcal{G}_1 \mathcal{G}_2 | \hat{\mathcal{Q}}_j | \mathcal{Q}_\omega(q_{(j+1)\perp}) \rangle g \, d_{i+1}(q_{(i+1)\perp}) + \langle \mathcal{G}_1 \mathcal{G}_2 | \bar{Q}_j \mathcal{Q}_{j+1} \rangle$$

$$= \langle \mathcal{G}_1 \mathcal{G}_2 | \mathcal{G}_\omega(q_{i\perp}) \rangle g \, \gamma^{Q_i}_{\mathcal{G}_i \mathcal{Q}_{i+1}} . \qquad (99)$$

It concludes the proof of the bootstrap relations.

7   Summary

The multi–Regge kinematics plays an outstanding role in high energy physics. It is extremely important since it gives a dominant contribution to total cross sections of particle interactions. The remarkable phenomenon is that QCD amplitudes in this kinematics have simple multi–Regge form and are expressed in terms of the gluon and quark Regge trajectories and a few vertices of Reggeon interactions.

The multi–Regge form of amplitudes containing quark exchanges was proposed in [2] long ago, but up to now it was merely tested on its self–consistency for several particular processes. In this paper we have presented the proof of the multi–Regge form in the leading logarithmic approximation for arbitrary quark–gluon inelastic processes in all orders of α_s. The proof is based on the bootstrap relations required by compatibility of the multi–Regge form (6) of inelastic QCD amplitudes with the s–channel unitarity.

It consists of three steps. First, we derive an infinite set of the bootstrap relations (21) and demonstrate that fulfillment of these relations secure the Reggeized form (6). Second, we show that all these bootstrap relations are fulfilled if the vertices and trajectories submit to several bootstrap conditions (50)–(52), (54) and (55). This circumstance is extremely nontrivial since an infinite set of the bootstrap relations is reduced to several conditions on the Reggeon trajectories and vertices. And finally, we examine the bootstrap conditions and prove that all of them are fulfilled.

Although being simple in principle, necessary calculations were extremely cumbersome and tedious if they performed in the standard approach. The operator formalism, recently introduced for consideration of elastic amplitudes with gluon exchanges and generalized in this paper for the case of inelastic amplitudes with arbitrary spin and colour exchanges, is very helpful.

References

[1] V.S. Fadin, E.A. Kuraev, and L.N. Lipatov, Phys. Lett. B **60** (1975) 50; E.A. Kuraev, L.N. Lipatov, and V.S. Fadin, Zh. Eksp. Teor. Fiz. **71** (1976) 840 [Sov. Phys. JETP **44** (1976) 443]; Zh. Eksp. Teor. Fiz. **72** (1977) 377 [Sov. Phys. JETP **45** (1977) 199].

[2] V.S. Fadin and V.E. Sherman, Pis'ma Zh. Eksp. Teor. Fiz. **23** (1976) 599 [Sov. Phys. JETP Lett. **23** (1976) 548]; Zh. Eksp. Teor. Fiz. **72** (1977) 1640 [Sov. Phys. JETP **45** (1977) 861].

[3] L.N. Lipatov and M.I. Vyasovsky, Nucl. Phys. B **597** (2001) 399, hep-ph/0009340.

[4] V.S. Fadin and R. Fiore, Phys. Rev. D **64** (2001) 114012, hep-ph/0107010.

[5] M.I. Kotsky, L.N. Lipatov, A. Principe, and M.I. Vyazovsky, Nucl. Phys. B **648** (2003) 277, hep-ph/0207169.

[6] A.V. Bogdan, V. Del Duca, V.S. Fadin, and E.W.N. Glover, JHEP **203** (2002) 32, hep-ph/0201240.

[7] A.V. Bogdan and V.S. Fadin, Yad. Phys. **9** 1659 (2005).

[8] Ya.Ya. Balitskii, L.N. Lipatov, and V.S. Fadin, in: Proceedings of XIV LIYAF Winter School; Leningrad, 1979, 109.

[9] V. S. Fadin, R. Fiore, M. I. Kotsky, and A. Papa, Phys. Lett. B **495** (2000) 329, hep-ph/0008057.

[10] L.N. Lipatov, Yad. Fiz. **23** (1976) 642.

[11] J. Bartels, Phys. Rev. D **11** 2977 (1975) 2989; Nucl. Phys. B **175** (1980) 365.

[12] V.S. Fadin, Talk given at the NATO Advanced Research Workshop "Diffraction 2002", August 31 - September 6, 2002, Alushta, Crimea, Ukraine, in Diffraction 2002, Ed. by R. Fiore et al., NATO Science Series, Vol. 101, p.235.

A.V. Bogdan, V.S. Fadin

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А.В. Богдан, В.С. Фадин

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