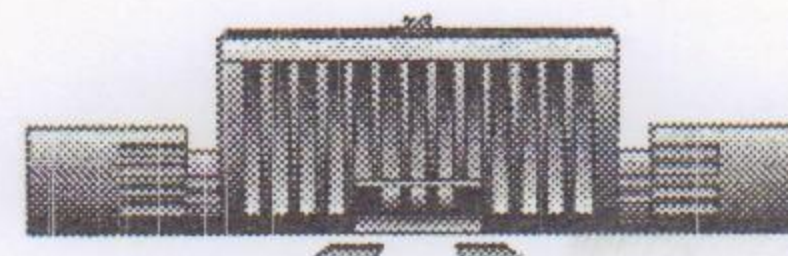V.N. Baier and V.M. Katkov

PAIR CREATION BY A PHOTON IN A STRONG MAGNETIC FIELD

Pair creation by a photon
in a strong magnetic field at high-energy

V.N. Baier[1] and V.M. Katkov

Institute of Nuclear Physics
630090, Novosibirsk, Russia

Abstract

The process of pair creation by a photon in a strong magnetic field is investigated basing on the polarization operator in the field. The total probability of the process is found in a relatively simple form. The probability exhibits a "saw-tooth" pattern because of divergences arising when the electron and positron are created at threshold of the Landau energy levels. The pattern will be washed out at averaging over any smooth photon energy distribution. The new results are obtained in the scope of the quasiclassical approach: 1) in the case when the magnetic field $B \ll B_0$, (B_0 is the critical field) the new formulation extends the photon energy interval to the case when the created particles are not ultrarelativistic; 2) the correction to the standard quasiclassical approximation is found showing the range of applicability of the approach at high photon energy as well. The very important conclusion is that for both cases $B \ll B_0$ and $B \geq B_0$ the results of the quasiclassical calculation are very close to averaged probabilities of exact theory in a very wide range of photon energies. The quasiclassical approximation is valid also for the energy distribution if the electron and positron are created on enough high levels.

[1]V.N.Baier@inp.nsk.su

1 Introduction

Study of QED processes in a strong external field is stimulated essentially by existence of very strong magnetic fields in nature. There is a series of evidences for the existence of neutron stars possessing magnetic fields close the critical field strength $B_0 = m^2/e = (m^2c^3/e\hbar) = 4.41 \cdot 10^{13}$ G. The rotating magnetic dipole model, in which the pulsar loses rotational energy through the magnetic dipole radiation, was confirmed with the discovery that the spin-down power predicted for many pulsars is in a quite good energetic agreement with the observed radiation, and gives magnetic fields $B \sim 10^{11} - 10^{13}$ G [1]. There are presently around 1600 spin-powered radio pulsars.

Another class of neutron stars was discovered at X-ray and γ-ray energies and may possess even stronger surface magnetic fields $B \sim 10^{14} - 10^{15}$ G. Such stars are now referred to as magnetars [2], since they most probably derive their power from their magnetic fields rather than from spin-down energy loss. Two different classes of objects are thought to be magnetars: the Soft Gamma-ray Repeaters (SGR), discovered as sources of short bursts of hard X-rays (> 30 keV) with super-Eddington luminosity and the Anomalous X-ray Pulsars (APS), discovered as persistent, soft (< 10 keV) X- ray sources with pulsations of several seconds and spinning-down on time scales of $10^4 - 10^5$ years. An increasing number of common properties, pointing to a close relationship of these two apparently different classes of objects and leading to their interpretation as magnetars, has been found (see e.g.[3]).

Pair creation by a photon in a strong magnetic field is the basic QED reaction which can play the significant role in processes in vicinity of a neutron star.

Investigation of pair creation by a photon in a strong magnetic field was started in 1952 independently by Klepikov and Toll [4, 5]. In Klepikov's paper [6], which was based on the solution of the Dirac equation in a constant and homogeneous magnetic field, the probabilities of radiation from an electron and e^-e^+ pair creation by a photon were obtained for magnetic field of arbitrary strength on the mass shell[2] ($k^2 = 0$, k is the 4-momentum of photon).

[2]We use the system of units with $\hbar = c = 1$ and the metric $ab = a^\mu b_\mu = a^0 b^0 - \mathbf{ab}$

In 1971 Adler [7] calculated the photon polarization operator in the mentioned magnetic field using the proper-time technique developed by Schwinger [8] and Batalin and Shabad [9] calculated the photon polarization operator in a constant and homogeneous electromagnetic field for $k^2 \neq 0$ using the Green function in this field found by Schwinger [8]. In 1975 Strakhovenko and present authors calculated the contribution of a charged-particles loop with n external photon lines having applied the proper-time method in a constant and homogeneous electromagnetic field [10]. The explicit expression for the photon polarization operator was found. In a purely magnetic field ($\mathbf{E}{=}0$) the imaginary part of the polarization operator, which determines the probability of the pair creation in general case ($k^2 \neq 0$), was analyzed in detail near the root singularities originating at each threshold of $e^- e^+$ pair creation in the state with the given Landau quantum numbers. This peculiarity of the pair creation process was observed by Klepikov [6]. Shabad [11] performed an extensive analysis of the photon polarization operator found in [9] in a purely magnetic field. It was shown that its imaginary part in the limit $k^2 \to 0$ reduces to the result of Klepikov's calculation [6]. Daugherty and Harding [12] analyzed in detail the total probabilities (the attenuation coefficients) of the pair creation by a polarized photon in the superstrong field $B \sim B_0$ basing on the calculations of Toll and Klepikov and paying the special attention to the threshold region $\omega \geq 2m$, where the probabilities exhibit a "saw-tooth" pattern due to the mentioned root singularities.

For high Landau levels Klepikov [6] (and Toll) found the quasiclassical representations for the probabilities of radiation and pair creation in the case $B \ll B_0$. In 1968 the present authors developed the operator quasiclassical method for the consideration of electromagnetic processes in an external field [13] which is valid also in nonuniform and (or) depending on time field under conditions: 1) $B \ll B_0$, 2) the charged particles have relativistic energies. Both the radiation and pair creation by a photon were considered. The method is given in detail in [14, 15]. Tsai and Erber [16] deduced the quasiclassical probability of pair creation using the photon polarization operator in a magnetic field calculated by Adler [7].

In the Sec.2 the exact probability of the pair creation by a photon was obtained for general case $k^2 \neq 0$ starting from the polarization operator in a magnetic field. The technical details of the derivation are given in Appendix A. In Sec.3 the standard quasiclassical approximation (SQA) is outlined in the limit $\mu \equiv B/B_0 \ll 1$, $r \equiv \omega^2/4m^2 \gg 1$, where ω is the photon energy. The quasiclassical characteristics of the process depend on the parameter $\kappa = 2\mu\sqrt{r}$. The correction to SQA is found. Details of calculation of the correction starting from the polarization operator in a magnetic field is given

in Appendix C. If the photon energy is not high, so that $\mu \ll r - 1 \ll \mu^{-2}$ and SQA is not applicable, the new quasiclassical approach (at low energy near threshold (th)) is developed in Appendix B. One can expect that the photon energy distribution in a vicinity of neutron star is wide and smooth. At averaging over this distribution the "saw-tooth" pattern will be washed out. It is very important that the result the averaging is very close to SQA even at $\mu \geq 1$. The corresponding analysis has been carried out in Sec.3. The energy spectrum of created pair is discussed in Sec.4. Here SQA is very useful also. A remark concerning derivation of the energy spectrum in SQA is given in Appendix D.

2 Probability of pair creation by a photon: exact theory

Our analysis is based on the expression for the polarization operator obtained in [10], see Eqs.(3.19), (3.33). For pure magnetic field ($\mathbf{E}{=}0$) this polarization operation can be written in the diagonal form

$$\Pi^{\mu\nu} = -\sum_i \kappa_i \beta_i^\mu \beta_i^\nu, \quad \beta_i \beta_j = -\delta_{ij}, \quad \beta_i k = 0, \quad \sum_i \beta_i^\mu \beta_i^\nu = \frac{k^\mu k^\nu}{k^2} - g^{\mu\nu},$$

(2.1)

where

$$\beta_1^\mu = \frac{k^2 k_\perp^\mu + \mathbf{k}_\perp^2 k^\mu}{k_\perp \sqrt{k^2(\omega^2 - k_3^2)}}, \quad \beta_2^\mu = \frac{F^{\mu\nu} k_\nu}{Bk_\perp}, \quad \beta_3^\mu = \frac{F^{*\mu\nu} k_\nu}{B\sqrt{\omega^2 - k_3^2}},$$

$$\kappa_1 = \Omega_1(r - q), \quad \kappa_2 = \kappa_1 - q\Omega_2, \quad \kappa_3 = \kappa_1 + r\Omega_3,$$

$$r = \frac{\omega^2 - k_3^2}{4m^2}, \quad q = \frac{k_\perp^2}{4m^2}, \quad r - q = \frac{k^2}{4m^2},$$

(2.2)

where the axis 3 directed along the magnetic field $\mathbf{B}$, $\mathbf{k}_\perp \mathbf{B} = 0$, $k_\perp = \sqrt{\mathbf{k}_\perp^2}$, ω is the photon energy, $F^{\mu\nu}$ is the tensor of electromagnetic field, $F^{*\mu\nu}$ is the dual tensor of electromagnetic field,

$$\Omega_i = -\frac{\alpha m^2}{\pi} \int_{-1}^{1} dv \int_0^{\infty - i0} f_i(v, x) \exp(i\psi(v, x)) dx.$$

(2.3)

Here

$$f_1(v,x) = \frac{\cos vx}{\sin x} - v\frac{\cos x \sin vx}{\sin^2 x}, \quad f_2(v,x) = 2\frac{\cos vx - \cos x}{\sin^3 x} - f_1(v,x),$$

$$f_3(v,x) = (1-v^2)\cot x - f_1(v,x), \quad \psi(v,x) = z\frac{\cos x - \cos vx}{\sin x} - b(v)x, \quad (2.4)$$

where

$$z = \frac{2q}{\mu}, \quad b(v) = \frac{1 - r(1-v^2)}{\mu}, \quad \mu = \frac{B}{B_0}. \quad (2.5)$$

Let us note that the integration contour in Eq.(2.3) is turned slightly down, and in the function Ω_1 in the integral over x the subtraction at $\mu = 0$ is implied.

The real part of the polarization operator defines the dispersive properties of the space region with magnetic field. At $r < 1$ ($b > 0$) one can turn the integration contour over x to the negative imaginary axis ($x \to -ix$) so that

$$\Omega_i = \frac{\alpha m^2}{\pi} \int_{-1}^{1} dv \int_0^{\infty} if_i(v,-ix)\exp\left(-bx - z\frac{\cosh x - \cosh vx}{\sinh x}\right)dx. \quad (2.6)$$

The functions Ω_i in Eq.(2.6) are real. These functions, studied in [17] at $\omega < 2m$ and arbitrary value of μ, define the index of refraction $n_{2,3}$. On mass shell $k^2 = 0$ one has

$$n_{2,3}^2 = 1 - \frac{\kappa_{2,3}}{\omega^2}. \quad (2.7)$$

For the mode 2 the photon polarization vector $\mathbf{e}_2$ is perpendicular to the plane formed by the vectors $\mathbf{B}$ and $\mathbf{k}$ and for mode 3 the photon polarization vector $\mathbf{e}_3$ lies in this plane ($n_2 \to n_\parallel$ and $n_3 \to n_\perp$ in Adler's notation [7]).

At $r > 1$ the polarization operator eigenvalue κ_i acquires an imaginary part, which determines the probability of e^-e^+ pair creation per unit length W_i by a photon with a given polarization

$$W = \frac{1}{\omega}\,\mathrm{Im}\,e^\mu e^{\nu *}\,\Pi_{\mu\nu}. \quad (2.8)$$

Details of calculation basing on Eqs.(2.2),(2.3) are given in Appendix A. Using $e_i^\mu = \beta_i^\mu$ we get the explicit expressions for W_i ($i = 1,2,3$) at $k^2 \neq 0$:

$$W_i = -\frac{\mathrm{Im}\,\kappa_i}{\omega} = \frac{2\alpha m\mu e^{-z}}{\omega\lambda_c\sqrt{r}}\sum_{n,m}\left(1 - \frac{\delta_{n0}}{2}\right)\vartheta(g)\frac{z^n k!}{\sqrt{g}\, l!}d_i, \quad (2.9)$$

where δ_{n0} is the Kronecker symbol, $\vartheta(g)$ is the Heaviside function: $\vartheta(g) = 1$ for $g \geq 0$, $\vartheta(g) = 0$ for $g < 0$,

$$l = \frac{m+n}{2}, \quad k = \frac{m-n}{2}, \quad g = r - 1 - m\mu + \frac{n^2\mu^2}{4r},$$

$$l_{max} = [d(r)], \quad d(r) = \frac{2(r - \sqrt{r})}{\mu}. \quad (2.10)$$

Here $[d(r)]$ is the integer part of $d(r)$,

$$d_1 = \left(\frac{r}{q} - 1\right)G, \quad d_2 = \frac{r}{q}G + 4\mu l\vartheta(k-1)L_{k-1}^{n+1}(z)L_k^{n-1}(z),$$

$$d_3 = \left(1 + \frac{m\mu}{2} - \frac{n^2\mu^2}{4r}\right)F + 2\mu l\vartheta(k-1)L_k^n(z)L_{k-1}^n(z),$$

$$G = \left(\frac{m\mu}{2} - \frac{n^2\mu^2}{4r}\right)F - 2\mu l\vartheta(k-1)L_k^n(z)L_{k-1}^n(z),$$

$$F = (L_k^n(z))^2 + \vartheta(k-1)\frac{l}{k}\left(L_{k-1}^n(z)\right)^2, \quad (2.11)$$

where $L_k^n(z)$ is the generalized Laguerre polynomial, z is defined Eq.(2.5). Obtained probabilities W_i agree with found by Shabad [11] but presented in much more compact form.

On the mass shell ($k^2 = 0$) in Eqs.(2.9)-(2.11) one has to put $r = q$. In this case $W_1 = 0$ and only two polarizations $i = 2$ and $i = 3$ remain. The probability of pair creation averaged over the photon polarizations acquires especially simple form (in the frame where $k_3 = 0$)

$$W = \frac{W_2 + W_3}{2} = \frac{\alpha e^{-z}}{\lambda_c}\sum_{n,m}\left(1 - \frac{\delta_{n0}}{2}\right)\vartheta(g)\frac{z^{n-1}k!}{\sqrt{g}\, l!}$$

$$\times\left[\left(1 + m\mu - \frac{n^2\mu}{z}\right)F + 4\mu l\vartheta(k-1)L_{k-1}^{n+1}(z)L_k^{n-1}(z)\right] \quad (2.12)$$

The probabilities of pair creation by a photon Eqs.(2.9), (2.12) contain the factor $1/\sqrt{g}$. The function $g = 0$, if

$$r = \left(\frac{\varepsilon(l) + \varepsilon(k)}{2m}\right)^2, \quad \varepsilon(l) = m\sqrt{1 + 2l\mu}, \quad (2.13)$$

where $\varepsilon(l)$ is the energy of charged particle on the Landau energy level. So the probability diverges when the electron and positron are created on the

Landau levels with electron and positron momentum $p_3 = 0$, see e.g. [6, 10]. The origin the singularity is due the properties of the space volume in the lowest order of perturbation theory (infinitesimally narrow level). The reason why this singularity is not a pole but a branch point is that motion along a field is not quantized.

3 Probability of pair creation by a photon: quasiclassical approximation

In the case $\mu \ll 1$, $r - 1 \gg 1$ ($l_{max} \gg 1$) the mentioned above divergences are situated very often: $\Delta l \sim \Delta r / \mu \sim 1$. On the photon energy scale the characteristic distance between the mentioned peaks will be $\Delta \omega \sim \omega_0$, where ω_0 is the frequency of motion of created particle in a magnetic field. This frequency defines the distance between energy levels in the quasiclassical approximation. So the resonance properties of pair creation probability become apparent if the width of energy levels and the effective spread of photon energy are small comparing with ω_0 value. In the opposite case the spectrum peculiarities will be washed out and the process probability will be quite adequate described by approximate expressions ultimately connected with quasiclassical character of created particle motion at large Landau quantum numbers.

When the magnetic field is weak comparing with critical one ($\mu \ll 1$) and for high energy photons ($r \gg 1$) the quasiclassical approximation was created in [6, 13, 16], where the process probabilities were derived using the different approaches. In [6] the asymptotic of Eq.(2.12) were obtained. The approach in [13] was developed for $\mu \ll 1$ and the particle motion was considered to be quasiclassical (and relativistic) from the very beginning; it is applicable for nonuniform (and non-stationary) fields. The pair creation probabilities in the form Eq.(C.6) (see Appendix C) were found in [16]. Below we will call it the standard quasiclassical approximation (SQA).

In Appendix C the corrections to SQA were calculated. For probability of pair creation by unpolarized photon the correction is

$$W^{(1)} = \frac{\alpha m^2 \mu^2}{60\sqrt{3}\pi\omega\kappa} \int_{-1}^{1} [2(1 + v^2 - 27z^2)K_{1/3}(z) + 3(7 - v^2)zK_{2/3}(z)] \frac{dv}{1 - v^2},$$

(3.1)

where

$$z = \frac{8}{3(1 - v^2)\kappa}, \quad \kappa = 2\sqrt{r}\mu.$$

(3.2)

At $\kappa \ll 1$ the main contribution is given by the term $\propto z^2$, it is

$$W^{(1)} = \frac{\sqrt{6}\alpha m^2 \mu^2}{5\omega\kappa^2} \exp\left(-\frac{8}{3\kappa}\right), \quad \frac{W^{(1)}}{W} = -\frac{32\mu^2}{15\kappa^3}.$$

(3.3)

It is seen that at $\kappa \ll 1$ the SQA is applicable if $\mu^2/\kappa^3 \ll 1$.

At $r \leq \mu^{-2/3}$ SQA becomes inapplicable. In this region one can use the process probability found in Appendix B, which is valid if $\mu \ll r - 1 \ll \mu^{-2}$. Using Eq.(B.5) we get for the probability of pair creation by unpolarized photon

$$W^{(th)} = \frac{\alpha m^2 \mu}{4\omega} \frac{3r - 1}{\sqrt{r(r - 1)l(r)\beta(r)}} \exp\left(-\frac{\beta(r)}{\mu}\right),$$

(3.4)

where

$$l(r) = \ln\frac{\sqrt{r} + 1}{\sqrt{r} - 1}, \quad \beta(r) = 2\sqrt{r} - (r - 1)l(r).$$

(3.5)

For large $r \gg 1$, limited by the condition $r \ll \mu^{-2}$ ($\kappa \ll 1$), we have

$$W^{(th)} \simeq \frac{3\alpha m^2 \mu}{8\omega} \sqrt{\frac{3r}{2}} \exp\left(-\frac{4}{3\mu\sqrt{r}} - \frac{4}{15\mu r^{3/2}}\right) = W^{(SQA)} \exp\left(-\frac{4}{15\mu r^{3/2}}\right)$$

(3.6)

At $\kappa^3 \gg \mu^2$ Eq.(3.6) agrees with Eq.(3.3). The last expression has essentially wider region of applicability than pure $W^{(SQA)}$.

The behavior of pair creation probability for unpolarized photon at $\mu = 0.1$ ($1/\mu \gg 1$) as a function of r is illustrated in Fig.1. The probability in saw-tooth form is calculated according to Eq.(2.12). It is seen that the characteristic distance between peaks is of the order $\mu = 0.1$. The result of averaging over this interval ($-\mu/2 + r \div \mu/2 + r$) is given by the dashed line which is smoothed out all peculiarities of original behavior and which is in a very good agreement with the thick curve calculated according to Eq.(3.4).

At $\kappa \geq 1$ Eq.(3.4) becomes inapplicable. However in this case SQA is valid. From Eq.(C.6) we have the probability of pair creation by unpolarized photon

$$W^{(SQA)} = \frac{\alpha m^2}{3\sqrt{3}\pi\omega} \int_{-1}^{1} \frac{9 - v^2}{1 - v^2} K_{2/3}(z)dv,$$
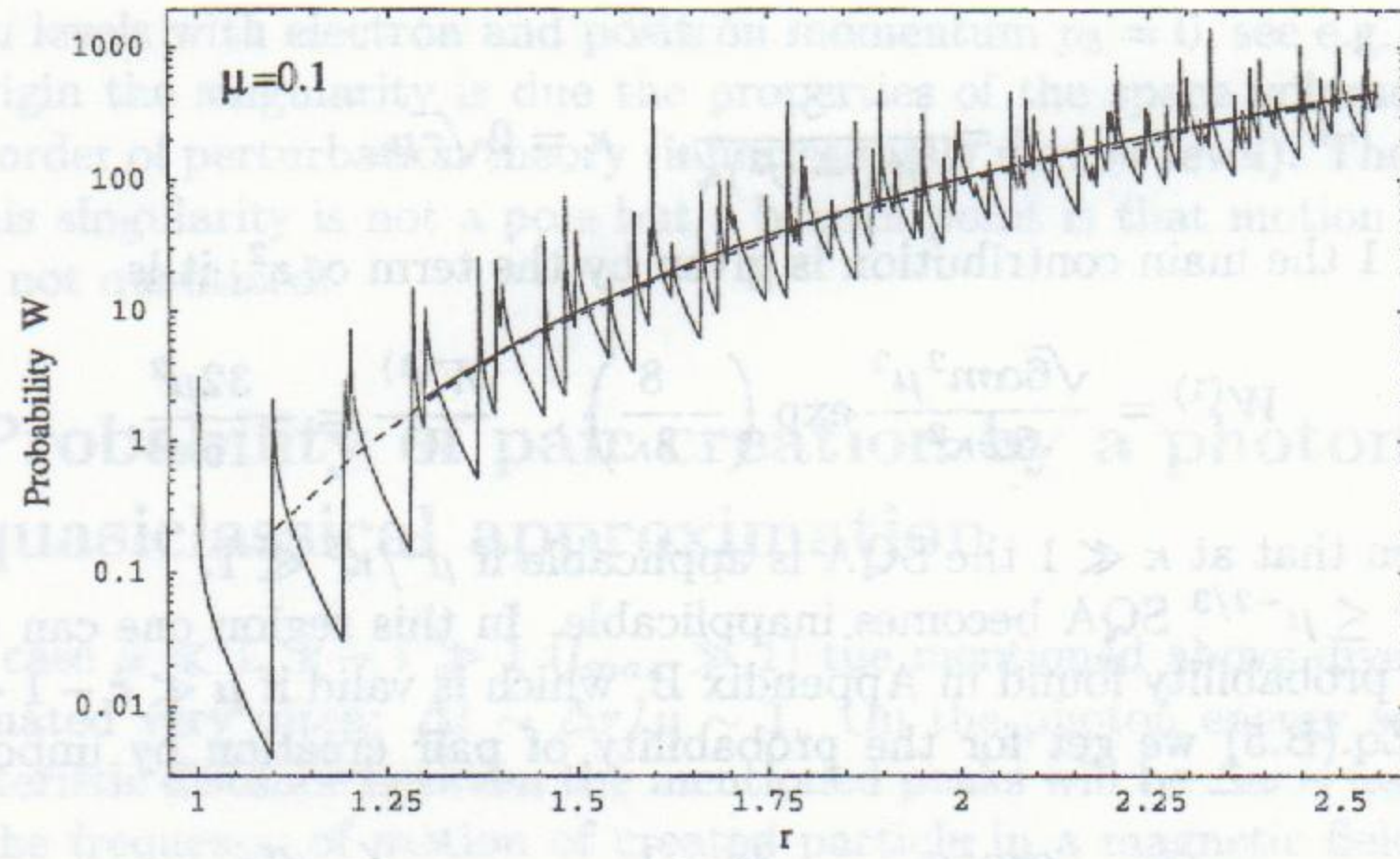
(3.7)

Figure 1: The pair creation probability for unpolarized photon at $\mu = 0.1$ as a function of r. The probability in saw-tooth form is calculated according to Eq.(2.12). The dashed line is the result of averaging over the interval $(-\mu/2 + r \div \mu/2 + r)$. The solid line is calculated according to Eq.(3.4).

where z is defined in Eq.(3.2). The correction to this probability is given by Eq.(3.1), its relative value at $\kappa \geq 1$ is $\leq \mu^2$. The ratio $W^{(th)}$ and $W^{(SQA)}$ is shown in Fig.2. The curves 2 and 3 are for polarized photons, the thick curves is for unpolarized photon (see Eq.(3.4) and Eq.(3.7)). The point where the ratio attains 1 is the boundary of the applicability region of the corresponding approximation.

The relative contribution of the correction Eq.(3.1) to SQA at $\mu = 3$ is shown in Fig.3. The upper curve is the probability of pair creation by a photon in SQA Eq.(3.7), The lower curve is the sum $W^{(SQA)} + W^{(1)}$ (the correction is negative at low r). It is seen that the correction is extraordinary small in the region of SQA applicability $r \gg \mu/2 = 1.5$.

It is very important that the quasiclassical approximation appears to be applicable also for super-strong magnetic field $\mu \geq 1$. This is true if the characteristic parameter $l_{max} \gg 1$. It is assumed in this case that parameter κ is large ($\kappa = 2\sqrt{r}\mu \gg 1$). In the limit $\kappa \gg 1$ the asymptotic expansion of the probability Eq.(3.7) is

$$W^{(SQA)} = \frac{\alpha m^2}{\pi \omega}\left(B\kappa^{2/3} - \frac{2\pi}{3} + \dots\right), \quad B = \frac{3^{7/6}5}{14}\frac{\Gamma^3(2/3)}{\Gamma(1/3)} = 1.1925.. \tag{3.8}$$
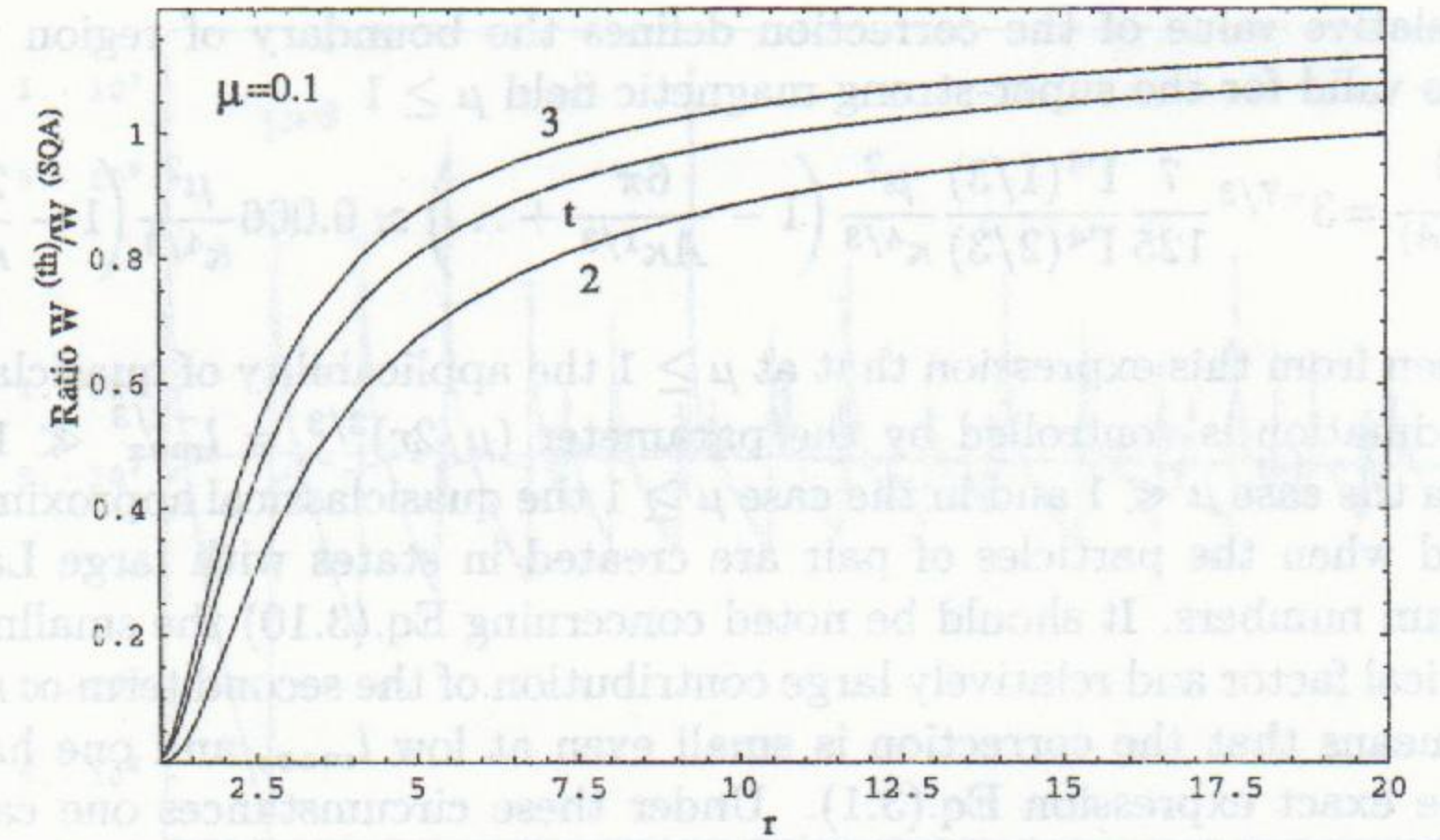

Figure 2: The ratio of probabilities $W^{(th)}/W^{(SQA)}$. The curves 2 and 3 are for the polarized photons, the curve t is for the unpolarized photon (see Eq.(3.4) and Eq.(3.7)).
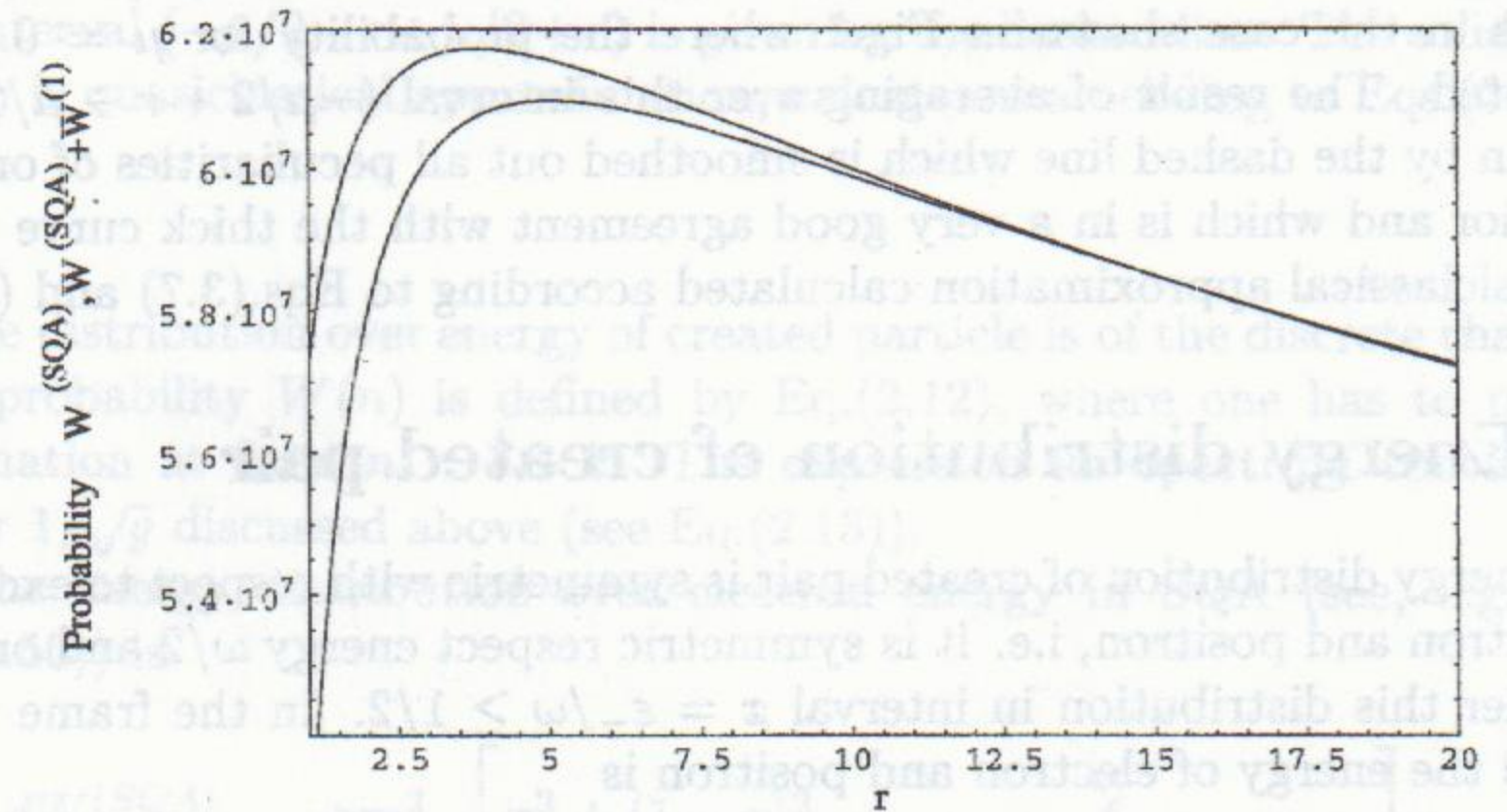

Figure 3: The pair creation by a photon probability at $\mu = 3$ in the quasiclassical approximation. The upper curve is the probability in SQA Eq.(3.7). The lower curve is the sum $W^{(SQA)} + W^{(1)}$ Eq.(3.1)

The asymptotic expansion of correction Eq.(3.1) is

$$W^{(1)} = \frac{\alpha m^2}{\pi \omega}\frac{\mu^2}{60\sqrt{3}\kappa}\left(A\kappa^{1/3} - 6\pi + \dots\right), \quad A = \frac{3^{1/3}2}{5}\frac{\Gamma^3(1/3)}{\Gamma(2/3)} = 8.191.. \tag{3.9}$$

The relative value of the correction defines the boundary of region where SQA is valid for the super-strong magnetic field $\mu \geq 1$

$$\frac{W^{(1)}}{W^{(SQA)}} = 3^{-7/3} \frac{7}{125} \frac{\Gamma^4(1/3)}{\Gamma^4(2/3)} \frac{\mu^2}{\kappa^{4/3}} \left(1 - \frac{6\pi}{A\kappa^{1/3}} + \ldots\right) \simeq 0.066 \frac{\mu^2}{\kappa^{4/3}} \left(1 - \frac{2.30}{\kappa^{1/3}}\right).$$
(3.10)

It is seen from this expression that at $\mu \geq 1$ the applicability of quasiclassical approximation is controlled by the parameter $(\mu/2r)^{2/3} = l_{max}^{-2/3} \ll 1$. So, both in the case $\mu \ll 1$ and in the case $\mu \geq 1$ the quasiclassical approximation is valid when the particles of pair are created in states with large Landau quantum numbers. It should be noted concerning Eq.(3.10) the smallness of numerical factor and relatively large contribution of the second term $\propto \kappa^{-1/3}$. This means that the correction is small even at low l_{max}, and one have to use the exact expression Eq.(3.1). Under these circumstances one can use Eq.(3.7) with the correction Eq.(3.1) starting from $r \sim \mu$. This is illustrated in Fig.4 where the probability of pair creation is shown at $\mu = 3$. The probability in saw-tooth form is calculated according to Eq.(2.12). It is seen that the characteristic distance between peaks is of the order μ (here $\mu = 3$) just as in the case shown in Fig.1 where the probability for $\mu = 0.1$ was presented. The result of averaging over this interval $(-\mu/2 + r \div \mu/2 + r)$ is given by the dashed line which is smoothed out all peculiarities of original behavior and which is in a very good agreement with the thick curve which is quasiclassical approximation calculated according to Eqs.(3.7) and (3.1).

4 Energy distribution of created pair

The energy distribution of created pair is symmetric with respect to exchange of electron and positron, i.e. it is symmetric respect energy $\omega/2$ and one can consider this distribution in interval $x = \varepsilon_-/\omega \geq 1/2$. In the frame where $k_3 = 0$ the energy of electron and positron is

$$\varepsilon_-(l) = \sqrt{p_3^2 + m^2(1 + 2\mu l)}, \quad \varepsilon_+(k) = \sqrt{p_3^2 + m^2(1 + 2\mu k)}, \quad \omega = \varepsilon_-(l) + \varepsilon_+(k).$$
(4.1)

From Eq.(4.1) follows that $p_3^2/m^2 = g$, (g is defined in Eq.(2.10)). In the plane (k, l) the allowed states of electron and positron are bounded by the axes and the curve $p_3^2 = 0$ which is parabola with the nearest point to origin $l_n = k_n = [(r-1)/2\mu]$. When $k = 0$ the maximal value $l = l_{max}$ (l_{max} is defined in Eq.(2.10)). Substituting p_3^2/m^2 in ε_- we get

$$x = \frac{\varepsilon_-}{\omega} = \frac{1}{2}\left(1 + \frac{n\mu}{2r}\right), \quad n = l - k.$$
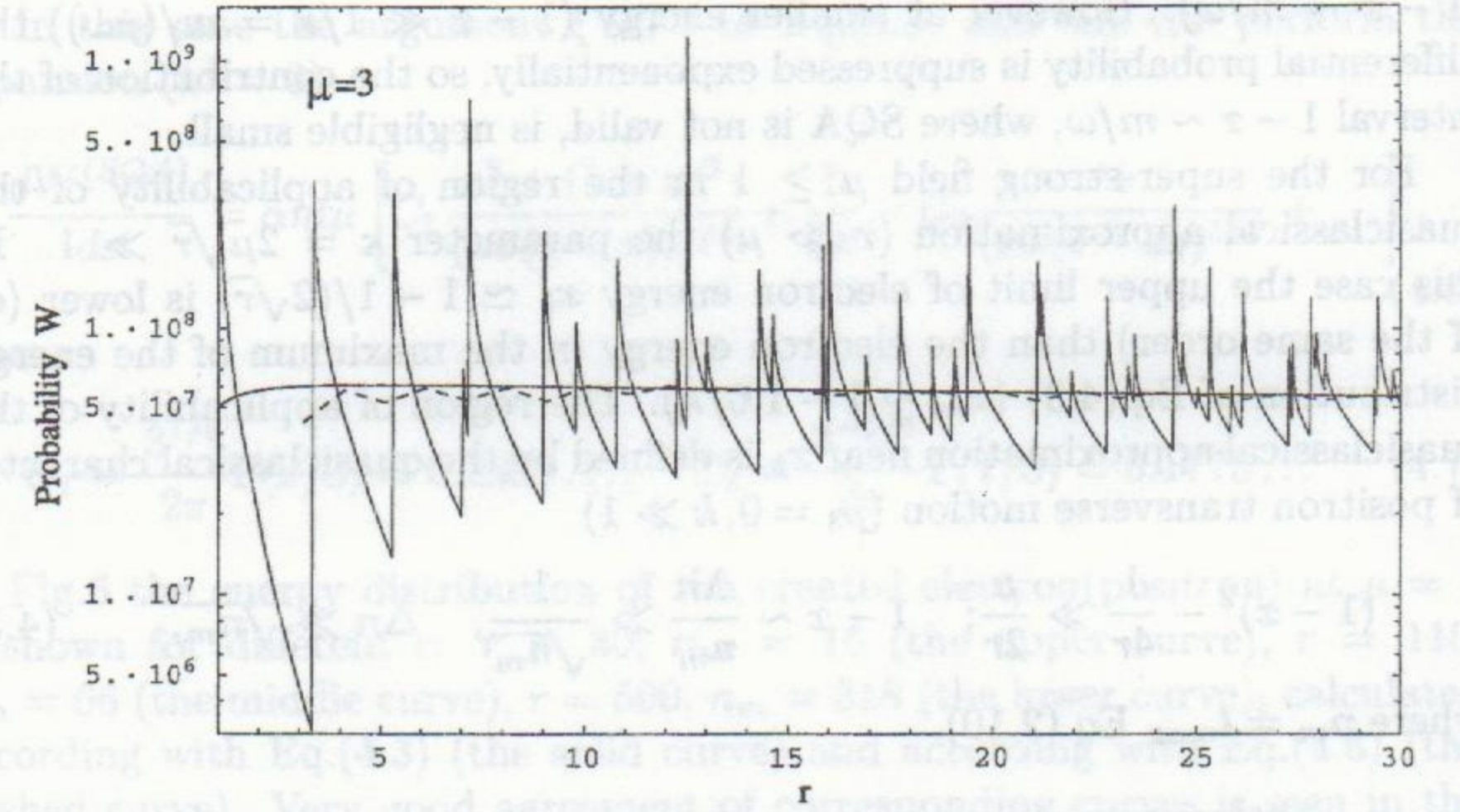(4.2)

Figure 4: The probability of pair creation at $\mu = 3$. The probability in saw-tooth form is calculated according to Eq.(2.12). The result of averaging over the interval $(-\mu/2 + r \div \mu/2 + r)$ is given by the dashed line. The solid curve, which is quasiclassical approximation, calculated according to Eqs.(3.7) and (3.1).

So the distribution over energy of created particle is of the discrete character. The probability $W(n)$ is defined by Eq.(2.12), where one has to perform summation at fixed $n = l - k$. The expression for spectrum contains the factor $1/\sqrt{g}$ discussed above (see Eq.(2.13)).

The energy distribution over electron energy in SQA (see, e.g. [15], Eq.(3.50)) is

$$\frac{dW^{(SQA)}}{dx} = \frac{\alpha m^2}{\sqrt{3}\pi\omega} \left[\frac{x^2 + (1-x)^2}{x(1-x)} K_{2/3}(\xi) + \int_\xi^\infty K_{1/3}(y)dy\right],$$
(4.3)

where

$$\xi = \frac{2}{3\kappa x(1-x)}.$$
(4.4)

At low values of the parameters μ and κ the energy distribution of pair creation probability has a sharp peak when $\varepsilon_- = \varepsilon_+ = \omega/2$ (for details see Sec.3.2 in [15]). When $\mu \ll 1$ and $\kappa \geq 1$ the SQA is not valid on the edge of the energy spectrum where another created particle is no more relativistic

$(1 - x \sim m/\omega)$. However at smaller energy $(1 - x \ll 1/\kappa = m/(\mu\omega))$ the differential probability is suppressed exponentially, so the contribution of the interval $1 - x \sim m/\omega$, where SQA is not valid, is negligible small.

For the super-strong field $\mu \geq 1$ in the region of applicability of the quasiclassical approximation $(r \gg \mu)$ the parameter $\kappa = 2\mu\sqrt{r} \gg 1$. In this case the upper limit of electron energy $x_b \simeq 1 - 1/(2\sqrt{r})$ is lower (or of the same order) than the electron energy in the maximum of the energy distribution of Eq.(4.3) $(x_m \sim 1 - 1.6/\kappa)$. The region of applicability of the quasiclassical approximation near x_b is defined by the quasiclassical character of positron transverse motion $(p_3 = 0, k \gg 1)$

$$(1 - x)^2 - \frac{1}{4r} \gg \frac{\mu}{2r}; \quad 1 - x \sim \frac{\Delta n}{n_m} \gg \frac{1}{\sqrt{n_m}}, \quad \Delta n \gg \sqrt{n_m}, \quad (4.5)$$
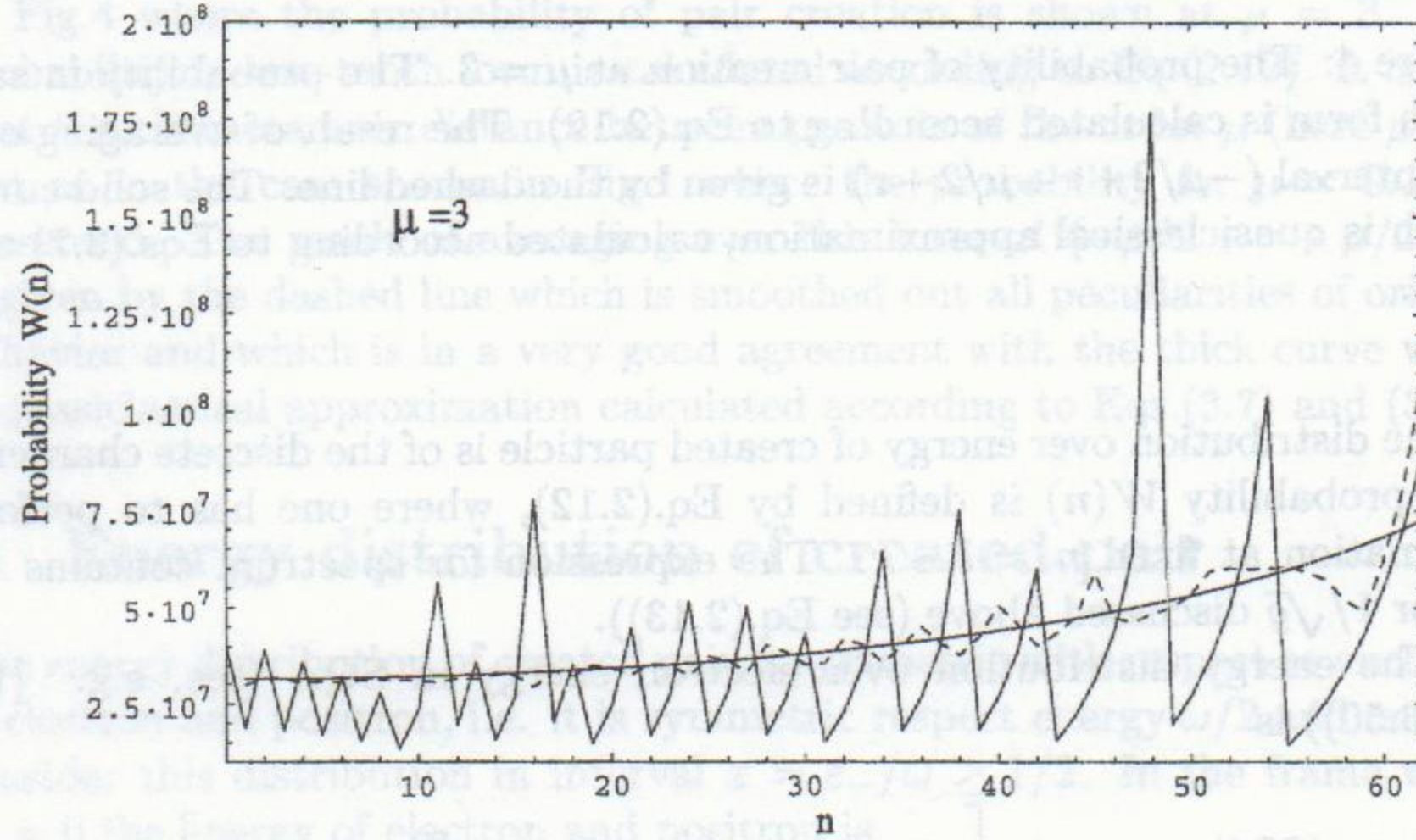
where $n_m = l_{max}$ Eq.(2.10).

Figure 5: The spectrum of created particles for $\mu = 3$, $r = 110$ $(n_m = 66)$ vs n (see Eq.(4.2)). The spectrum in saw-tooth form is calculated according to Eq.(2.12). The dashed curve is the spectrum averaged over the interval $(-2\mu + r \div 2\mu + r)$. The solid curve calculated according to Eq.(4.3).

In Fig.5 the spectrum of created particles is shown for $\mu = 3$, $r = 110$ $(n_m = 66)$. The spectrum in saw-tooth form is calculated according to Eq.(2.12). The dashed curve is the spectrum averaged over the interval $(-2\mu + r \div 2\mu + r)$. The thick curve calculated using Eq.(4.3). It is seen that this curve is in a quite good agreement with the result of averaging.

In this case the argument $\xi \ll 1$ in Eq.(4.3) and one can perform the expansion $(k^3 = 0)$

$$\frac{dW^{(SQA)}}{dx} = \alpha m\mu \left[C_1 \frac{x^2 + (1-x)^2}{(\kappa x(1-x))^{1/3}} + \frac{1}{3\kappa} - C_2 \frac{1}{(\kappa x(1-x))^{5/3}} + \dots \right],$$
$$(4.6)$$

where

$$C_1 = \frac{3^{1/6}}{2\pi}\Gamma(2/3) = 0.2588\dots, \quad C_2 = \frac{3^{-1/6}}{4\pi}\Gamma(1/3) = 0.1775\dots \quad (4.7)$$

In Fig.6 the energy distribution of the created electron(positron) at $\mu = 3$ is shown for different r: $r = 30$, $n_m = 16$ (the upper curve), $r = 110$, $n_m = 66$ (the middle curve), $r = 500$, $n_m = 318$ (the lower curve), calculated according with Eq.(4.3) (the solid curve) and according with Eq.(4.6) (the dashed curve). Very good agreement of corresponding curves is seen in the region of SQA applicability $(x \geq n_m^{-1/2}, (1 - x) \geq n_m^{-1/2})$.
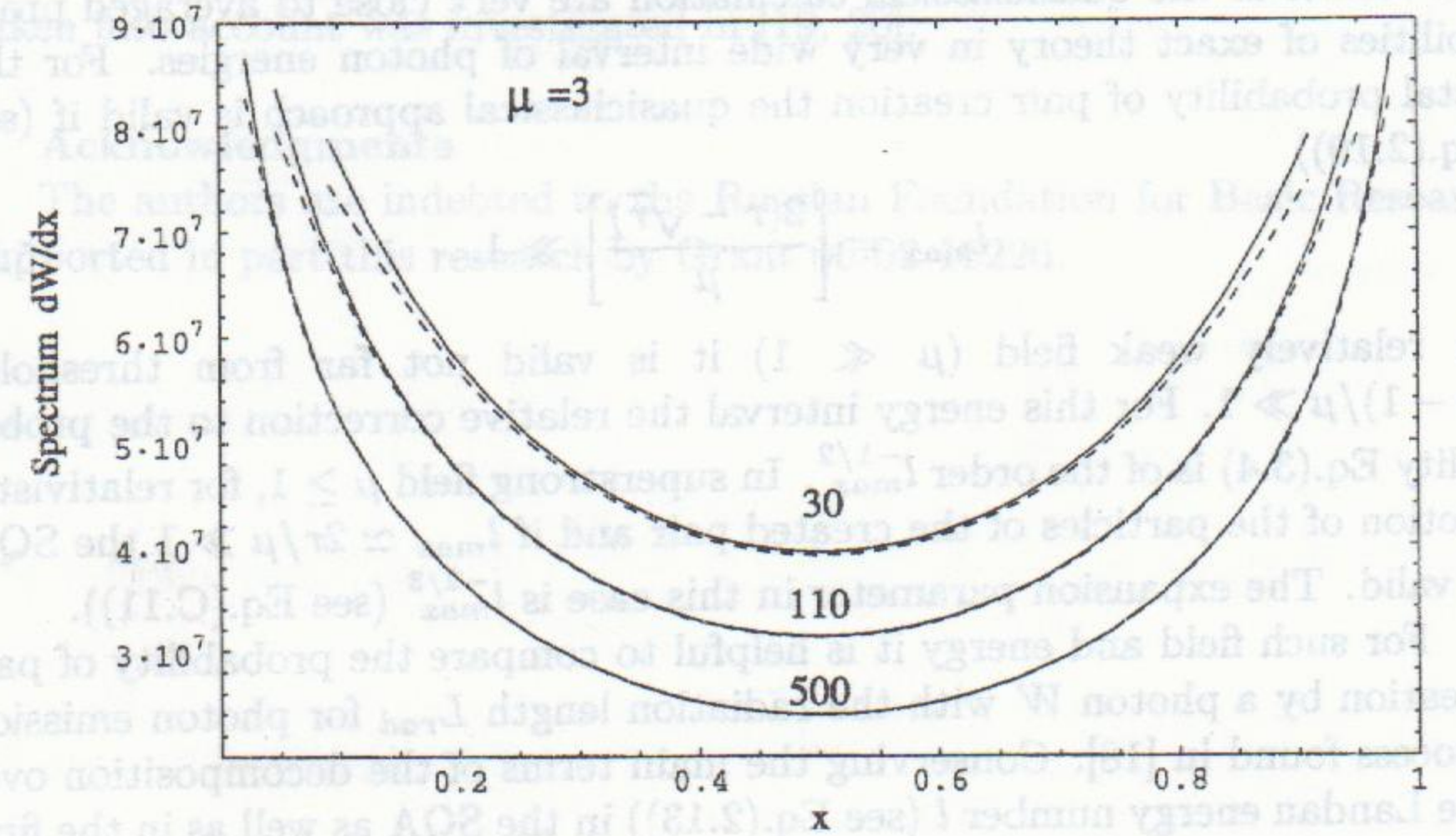

Figure 6: The energy distribution of the created electron(positron) at $\mu = 3$ for different r: $r = 30$, $n_m = 16$ (the upper curve), $r = 110$, $n_m = 66$ (the middle curve), $r = 500$, $n_m = 318$ (the lower curve), calculated according to Eq.(4.3) (the solid curve) and according to Eq.(4.6) (the dashed curve).

5 Conclusion

In this paper we study the process of pair creation by a photon in a strong magnetic field basing on the polarization operator in the field calculated by different methods in a set of papers [7, 9, 10]. By using the decomposition of functions into series containing the Bessel function (see Eq.(A.4)) we deduced the imaginary part of the polarization operator (which is the process probability) into the relatively simple combination of the generalized Laguerre polynomial (see Eq.(2.12)). This probability exhibits a "saw-tooth" pattern because of the factor $1/\sqrt{g}$. This pattern will be washed out at averaging over any smooth photon energy distribution.

The range of applicability of the quasiclassical approximation extended in the photon energy interval where the created particles are no more ultrarelativistic (see Appendix B). Found correction to standard quasiclassical approximation (SQA) permitted to extend the range of applicability of the quasiclassical approximation to the region $r \sim \mu$ at $\mu \geq 1$. From the performed analysis follows the remarkable conclusion: for both $\mu \ll 1$ and $\mu \geq 1$ the results of the quasiclassical calculation are very close to averaged probabilities of exact theory in very wide interval of photon energies. For the total probability of pair creation the quasiclassical approach is valid if (see Eq.(2.10))

$$l_{max} = \left[\frac{2(r - \sqrt{r})}{\mu} \right] \gg 1. \tag{5.8}$$

In relatively weak field ($\mu \ll 1$) it is valid not far from threshold: $(r-1)/\mu \gg 1$. For this energy interval the relative correction to the probability Eq.(3.4) is of the order $l_{max}^{-1/2}$. In superstrong field $\mu \geq 1$, for relativistic motion of the particles of the created pair and if $l_{max} \simeq 2r/\mu \gg 1$ the SQA is valid. The expansion parameter in this case is $l_{max}^{-2/3}$ (see Eq.(C.11)).

For such field and energy it is helpful to compare the probability of pair creation by a photon W with the radiation length L_{rad} for photon emission process found in [18]. Conserving the main terms of the decomposition over the Landau energy number l (see Eq.(2.13)) in the SQA as well as in the first correction we get ($k^3 = 0$)

$$L_{rad}^{-1} = \frac{a_\gamma}{L} \left[\frac{1}{l^{1/6}} + \frac{b_\gamma}{l^{5/6}} \right], \quad W = \frac{a_p}{L} \left[\frac{1}{l_{max}^{1/6}} + \frac{b_p}{l_{max}^{5/6}} \right], \tag{5.9}$$

where

$$\frac{1}{L} = \alpha^{3/2} \left(\frac{B}{e} \right)^{1/2}, \quad \alpha = \frac{e^2}{\hbar c} = \frac{1}{137}, \tag{5.10}$$

$$a_\gamma = \frac{32}{81}(18)^{-1/6}\Gamma(2/3) = 0.33045\ldots, \quad a_p = \frac{15}{7}(18)^{-1/6}\frac{\Gamma^2(2/3)}{\Gamma^2(1/3)} = 0.33819\ldots,$$

$$b_\gamma = \frac{(12)^{-1/3}}{20}\frac{\Gamma(1/3)}{\Gamma(2/3)} = 0.043206\ldots, \quad b_p = \frac{7(12)^{-1/3}}{1125}\frac{\Gamma^4(1/3)}{\Gamma^4(2/3)} = 0.041633\ldots,$$

where the numbers are taken from Eq.(46) in [18] and Eqs.(3.8)), (3.9)) above. It should be noted that the physics characteristics in Eq.(5.9)) don't depend on the electron mass. It's important that in the formula not only coefficients a_γ and a_p are very close, but the same properties possess the coefficients b_γ and b_p reflecting proximity of details of mechanisms of radiation and pair creation by a photon in a magnetic field. One can expect that this property will be essential for consideration of electron-photon shower in a magnetic field. Set of kinetic equations for such shower with the particle polarization taken into account was investigated in [19, 20].

A   Appendix

Let us consider the integral in Eq.(2.3)

$$T_i = \int_{-1}^{1} dv \int_{0}^{\infty - i0} f_i(v, x) \exp(i\psi(v, x)) dx. \qquad (A.1)$$

Using the formula

$$\frac{2}{iz} \cot x \frac{d}{dx} e^{i\psi} = \left(f_2 - f_1 - \frac{2b}{z} \cot x \right) e^{i\psi}, \qquad (A.2)$$

and integrating by parts the integral over x in Eq.(A.1) we get

$$\int_{0}^{\infty - i0} f_2 e^{i\psi} dx = \int_{0}^{\infty - i0} f_1 e^{i\psi} dx + \frac{2}{z} \int_{0}^{\infty - i0} \left(b \cot x - \frac{i}{\sin^2 x} \right) e^{i\psi} dx. \qquad (A.3)$$

Using Eq.7.2.4.(27) [21] we have

$$e^{-it\cos vx} = \sum_{n=0}^{\infty} (2 - \delta_{n0})(-i)^n J_n(t) \cos(nvx), \quad t = \frac{z}{\sin x},$$

$$e^{-it\cos vx} \cos vx = i \frac{d}{dt} e^{-it\cos vx}$$

$$= \sum_{n=0}^{\infty} \left(1 - \frac{\delta_{n0}}{2} \right)(-i)^{n+1} (J_{n+1}(t) - J_{n-1}(t)) \cos(nvx),$$

$$e^{-it\cos vx} \sin vx = \frac{1}{itv} \frac{d}{dx} e^{-it\cos vx} = \frac{2i}{t} \sum_{n=1}^{\infty} (-i)^n n J_n(t) \sin(nvx), \qquad (A.4)$$

where $J_n(t)$ is the Bessel function. Taking into account that the functions f_i and ψ in the integral over v in Eq.(2.3) are the even functions of v, we can perform following substitutions in sums in Eq.(A.4)

$$\cos(nvx) \to e^{invx}, \quad \sin(nvx) \to -ie^{invx}. \qquad (A.5)$$

So the expression for T_i Eq.(A.1) can be written as

$$T_i = \sum_n T_i^{(n)}, \quad T_i^{(n)} = \int_{-1}^{1} dv \int_{0}^{\infty - i0} F_i^{(n)}(v, x) \exp(ia_n(v)x) dx, \qquad (A.6)$$

where

$$F_1^{(n)} = (-i)^n \exp(iz \cot x) \left[\frac{i}{\sin x} \left(1 - \frac{\delta_{n0}}{2} \right) (J_{n+1}(t) - J_{n-1}(t)) \right.$$
$$\left. - \frac{2vn}{z} \cot x J_n(t) \right],$$

$$F_2^{(n)} = F_1^{(n)} + (-i)^n \exp(iz \cot x) \frac{2(2 - \delta_{n0})}{z} \left(b \cot x - \frac{i}{\sin^2 x} \right) J_n(t),$$

$$F_3^{(n)} = (-i)^n \exp(iz \cot x)(1 - v^2) \cot x (2 - \delta_{n0}) J_n(t) - F_1^{(n)},$$

$$a_n(v) = nv - b, \qquad (A.7)$$

and the function $F_i^{(n)}$ is a periodic function of x.

Let us note that at $x \to -i\infty$ the asymptotic of $J_n(z/\sin x)$ is

$$J_n \left(\frac{iz}{\sinh x} \right) \simeq J_n(2ize^{-x}) \simeq \frac{(iz)^n}{n!} e^{-nx}. \qquad (A.8)$$

In the case $a_n(v) < n$ the integration contour over x in Eq.(A.6) can be unrolled to the lower imaginary semiaxis and the contribution to the imaginary part of $T_i^{(n)}$ vanishes.

One can represent $T_i^{(n)}$ in the form

$$T_i^{(n)} = \int_{-1}^{1} dv \int_{0}^{2\pi} F_i^{(n)}(v, x) \exp(ia_n(v)x) \sum_{k=0}^{\infty} e^{2\pi i k a_n(v)} dx$$

$$= \int_{-1}^{1} \frac{dv}{1 - e^{2\pi i a_n(v)} + i0} \int_{0}^{2\pi} F_i^{(n)}(v, x) \exp(ia_n(v)x) dx. \qquad (A.9)$$

We will use in Eq.(A.9) the known equality

$$\frac{1}{1 - e^{2\pi i a_n(v)} + i0} = \frac{\mathcal{P}}{1 - e^{2\pi i a_n(v)}} - i\pi \delta(1 - e^{2\pi i a_n(v)}). \qquad (A.10)$$

By virtue of the made above observation

$$-i\pi\delta(1 - e^{2\pi i a_n(v)}) = -i\pi \sum_m \delta(1 - e^{2\pi i(a_n(v) - m)}) \to \frac{1}{2} \sum_{m \geq n} \delta(a_n(v) - m). \qquad (A.11)$$

Substituting Eq.(A.10) and Eq.(A.11) into Eq.(A.9) and taking into account that

$$F_i^{(n)}(v, x + \pi) = (-1)^n F_i^{(n)}(v, x), \qquad (A.12)$$

we have

$$T_i^{(n)} = (-1)^n \frac{i}{2} \mathcal{P} \int_{-1}^{1} \frac{dv}{\sin(\pi a_n(v))} \int_{-\pi}^{\pi} F_i^{(n)}(v, x) \exp(i a_n(v) x) dx$$

$$+ \sum_{m \geq n} \sum_{v_{1,2}} \frac{1 + (-1)^{n+m}}{2|a_n'(v)|} \int_0^{\pi} F_i^{(n)}(v_{1,2}, x) \exp(imx) dx, \qquad (A.13)$$

where

$$v_{1,2} = \frac{n\mu}{2r} \pm \sqrt{\frac{n^2 \mu^2}{4r^2} + 1 - \frac{1 + m\mu}{r}},$$

$$|a_n'(v)| = \frac{2}{\mu} \sqrt{rg(n, m, r)}, \quad g(n, m, r) = r - 1 - m\mu + \frac{n^2 \mu^2}{4r}. \quad (A.14)$$

It is seen from Eq.(A.13) that $n + m$ is even. Using Eq.(A.12) one can represent the imaginary part of $T_i^{(n,m)}$ in the form

$$\mathrm{Im} T_i^{(n,m)} = -\frac{i}{2} \sum_{v_{1,2}} \frac{\mu}{\sqrt{rg(n, m, r)}} \int_{-\pi/2}^{\pi/2} F_i^{(n)}(v_{1,2}, x) \exp(imx) dx. \quad (A.15)$$

Along with integers m and n we will use also $l = (m+n)/2$ and $k = (m-n)/2$.

We turn now to calculation of integrals in Eq.(A.15). Let us consider the integral

$$C_{n+1}^m(z) = (-i)^{n+1} \int_{-\pi/2}^{\pi/2} J_{n+1}\left(\frac{z}{\sin x}\right) e^{i(mx + z \cot x)} \frac{dx}{\sin x}. \quad (A.16)$$

Changing the variable $t = \cot x$ we get

$$C_{n+1}^m(z) = (-i)^{n+1} \int_{-\infty}^{\infty} J_{n+1}\left(z\sqrt{t^2 + 1}\right) \left(\frac{t + i}{t - i}\right)^{m/2} e^{izt} \frac{dt}{\sqrt{t^2 + 1}}. \quad (A.17)$$

One can close the integration contour in Eq.(A.17) by an infinite half-circle in the upper half-plane and than contract the contour to the point $t = i$. So, we have to calculate the integral in Eq.(A.17) over the contour $(i+)$. Changing successive the variables $t \to it$, $y = (t-1)/(t+1)$ we get

$$C_{n+1}^m(z) = e^{-z} \oint_{(0+)} I_{n+1}\left(\frac{2z\sqrt{y}}{1-y}\right) y^{-m/2} e^{-2zy/(1-y)} \frac{dy}{\sqrt{y}(1-y)}. \quad (A.18)$$

Using Eq.10.12.(20) [21]

$$\frac{1}{1-y} e^{-2zy/(1-y)} (z\sqrt{y})^{-n} I_n\left(\frac{2z\sqrt{y}}{1-y}\right) = \sum_{k=0}^{\infty} \frac{k!}{(k+n)!} (L_k^n(z))^2 y^k, \quad (A.19)$$

where $I_n(x)$ is the modified Bessel function, $L_k^n(z)$ is the is the generalized Laguerre polynomial, we get

$$C_{n+1}^m(z) = 2\pi i e^{-z} z^{n+1} \frac{(k-1)!}{l!} (L_{k-1}^{n+1}(z))^2 \vartheta(k-1), \quad n = l - k. \quad (A.20)$$

In the same manner one can calculate the integral

$$S_n^m(z) = 2(-i)^n \int_{-\pi/2}^{\pi/2} \cot x J_n\left(\frac{z}{\sin x}\right) e^{i(mx + z \cot x)} dx = 2\pi i e^{-z} z^n \frac{k!}{l!} F(l, k, z),$$

$$F(l, k, z) = (L_k^n(z))^2 + \frac{l}{k} (L_{k-1}^n(z))^2 \vartheta(k-1). \qquad (A.21)$$

Somewhat different structure has the integral

$$D_n^m(z) = (-i)^n \int_{-\pi/2}^{\pi/2} J_n\left(\frac{z}{\sin x}\right) e^{i(mx + z \cot x)} \frac{dx}{\sin^2 x}$$

$$= 2i e^{-z} \oint_{(0+)} I_{n+1}\left(\frac{2z\sqrt{y}}{1-y}\right) y^{-m/2} e^{-2zy/(1-y)} \frac{dy}{(1-y)^2}$$

$$=, 2i e^{-z} z^n \oint_{(0+)} \sum_{s=0}^{\infty} \sum_{j=0}^{\infty} \frac{j!}{(j+n)!} (L_j^n(z))^2 y^{j+s-k}$$

$$= -4\pi e^{-z} z^n \sum_{j=0}^{k-1} \frac{j!}{(j+n)!} (L_j^n(z))^2. \qquad (A.22)$$

Using Eq.10.12.(9) [21]:

$$\sum_{j=0}^{k-1} \frac{j!}{(j+n)!} L_j^n(x) L_j^n(y) = \frac{k!}{(k+n-1)!} \frac{1}{x-y} [L_{k-1}^n(x) L_k^n(y) - L_k^n(x) L_{k-1}^n(y)],$$
(A.23)

one obtains

$$D_n^m(z) = -4\pi e^{-z} z^n \frac{k!}{(l-1)!} \left[L_k^n(z) \frac{d}{dz} L_{k-1}^n(z) - L_{k-1}^n(z) \frac{d}{dz} L_k^n(z) \right]$$

$$= 4\pi e^{-z} z^n \frac{k!}{l!} l \left[L_{k-1}^{n+1}(z) L_k^{n-1}(z) - L_k^n(z) L_{k-1}^n(z) \right].$$
(A.24)

In derivation of the last expression Eqs.10.12.(15),(16),(24) [21] was used. In conclusion of this Appendix let us note, that $T_i^{(n,m)}$ contains the combination

$$C_{n+1}^m(z) + C_{n-1}^m(z) = 2\pi i e^{-z} z^{n-1} \frac{k!}{l!} \left[\frac{z^2}{k} (L_{k-1}^{n+1}(z))^2 + l(L_k^{n-1}(z))^2 \right]$$

$$= 2\pi i e^{-z} z^{n-1} \frac{k!}{l!} \left[\frac{1}{k} (l L_{k-1}^n(z) - k L_k^n(z))^2 + l(L_k^n(z) - L_{k-1}^n(z))^2 \right]$$

$$= 2\pi i e^{-z} z^{n-1} \frac{k!}{l!} \left[m F(l,k,z) - 4l L_k^n(z) L_{k-1}^n(z) \right].$$
(A.25)

It should be noted that the above derivation is the most direct path from the process probability in the form of the integral Eqs.(2.2),(2.3) to the process probability in the form Eq.(2.9).

B Appendix
Quasiclassical approximation at low photon energy
($\omega \sim m$, $k^2 = 0$, $r = q$)

In the field which is weak comparing with the critical field $B/B_0 = \mu \ll 1$ and at relatively low photon energy ($r \sim 1$) the created particles occupy mainly states with high quantum numbers if the condition $r - 1 \gg \mu$ is fulfilled. In this case the quasiclassical approach is valid, but created particles are no more ultrarelativistic, so the standard quasiclassical formulas [13, 14, 15] are nonapplicable. We will develop here another approach, using the method of stationary phase at calculation of the imaginary part of the integral over x in Eq.(2.3). Granting that the large parameter $1/\mu$ is the common factor in the phase $\psi(x)$ and doesn't contained in the equation $\psi'(x) = 0$

which defines the stationary phase point $x_0(r \sim 1) \sim 1$. In this case the small values of variable v contribute to the integral over v, so that one can extend the integration limits to the infinity. So we get

$$\mathrm{Im}\,\Omega_i \simeq i \frac{\alpha m^2}{2\pi} \int_{-\infty}^{\infty} dv \int_{-\infty}^{\infty} f_i(v,x) \exp\left\{ -\frac{i}{\mu} [\varphi(x) + v^2 \chi(x)] \right\} dx,$$
(B.1)

where

$$\varphi(x) = 2r \tan\left(\frac{x}{2}\right) + (1-r)x, \quad \chi(x) = rx\left(1 - \frac{x}{\sin x}\right).$$
(B.2)

From the equation $\varphi'(x) = 0$ we find

$$\tan\left(\frac{x_0}{2}\right) = -\frac{i}{\sqrt{r}}, \quad x_0(r) = -il(r), \quad l(r) = \ln\frac{\sqrt{r}+1}{\sqrt{r}-1}.$$
(B.3)

Substituting these results in the expressions which defines the integral in Eq.(B.1) we have

$$i\varphi(x_0) = \beta(r) = 2\sqrt{r} - (r-1)l(r), \quad i\varphi''(x_0) = \frac{r-1}{\sqrt{r}}, \quad i\chi(x_0) = \frac{\sqrt{r}}{2}l(r)\beta(r),$$

$$if_2(v=0, x_0(r)) = \frac{r-1}{2r^{3/2}}, \quad -if_3(v=0, x_0(r)) = \frac{1}{\sqrt{r}}.$$
(B.4)

Performing the standard procedure of the method of stationary phase one obtains for probability of pair creation by a polarized photon not far from threshold

$$W_3^{(th)} = \frac{\alpha m^2 \mu}{\omega} \sqrt{\frac{r}{(r-1)l(r)\beta(r)}} \exp\left(-\frac{\beta(r)}{\mu}\right), \quad W_2^{(th)} = \frac{r-1}{2r} W_3^{(th)}.$$
(B.5)

In spite of made above assumption $r \sim 1$ Eq.(B.5) is valid also at $r \gg 1$ if the condition $\beta(r) \gg \mu$ is fulfilled. This can be traced in the derivation of Eq.(B.5). The first two term of the decomposition of the function $\beta(r)$ over power of $1/r$ are

$$\frac{\beta(r)}{\mu} \simeq \frac{4}{3\mu\sqrt{r}} + \frac{4}{15\mu r^{3/2}}$$
(B.6)

It follows from this formula that that applicability of Eq.(B.5) is limited by the condition $r \ll \mu^{-2}$. If the second term much smaller than unity the

exponent with it can be expanded. As a result we have from Eq.(B.5) at $\mu^{-2/3} \ll r \ll \mu^{-2}$

$$W_3 = \frac{\alpha m^2 \mu}{2\omega} \sqrt{\frac{3r}{2}} \exp\left(-\frac{4}{3\mu\sqrt{r}}\right)\left(1 - \frac{4}{15\mu r^{3/2}}\right), \quad W_2 = \frac{1}{2}W_3 \quad \text{(B.7)}$$

The main term in above expression coincides with the probability of pair creation by a photon in standard quasiclassical theory at $\kappa = 2\mu\sqrt{r} \ll 1$. The correction in Eq.(B.7) determines the lower boundary over the photon energy of standard approach applicability ($\kappa^3 \gg \mu^2$). So exists the overlapping region where both the formulated here and the standard approach for high energy are valid.

C Appendix

Corrections to the standard quasiclassical approximation ($k^2 = 0$, $r = q$)

The standard quasiclassical approximation valid for ultrarelativistic created particles ($r \gg 1$) can be derived from Eqs.(2.3), (2.4) by expanding the functions $f_2(v,x), f_3(v,x), \psi(v,x)$ over x powers. Taking into account the higher powers of x one gets

$$f_2(v,x) = \frac{1-v^2}{12}\left[(3+v^2)x + \frac{1}{15}(15 - 6v^2 - v^4)x^3\right]$$

$$f_3(v,x) = -\frac{1-v^2}{6}\left[(3-v^2)x + \frac{1}{60}(15 - 2v^2 + 3v^4)x^3\right]$$

$$\psi(v,x) = -\frac{r(1-v^2)^2}{12\mu}\left(x^3 + \frac{3-v^2}{30}x^5\right) - \frac{x}{\mu}. \quad \text{(C.1)}$$

Here the first terms in the brackets give the known probability of the process in the standard quasiclassical approximation, while the second terms are the corrections. Expanding the term with x^5 in $\exp(i\psi(v,x))$ and making substitution $x = \mu t$ one finds

$$\text{Im}\,\Omega_i = i\frac{\alpha m^2 \mu}{2\pi}\int_{-1}^{1} dv \int_{-\infty}^{\infty} g_i(v,t)\exp\left[-i\left(t + \xi\frac{t^3}{3}\right)\right]dx, \quad \text{(C.2)}$$

$$g_2(v,t) = \frac{1-v^2}{12}\mu t\left[(3+v^2) - i\frac{9-v^4}{90}\xi\mu^2 t^5 + \frac{\mu^2 t^2}{15}(15 - 6v^2 - v^4)\right],$$

$$g_3(v,t) = -\frac{1-v^2}{6}\mu t\left[(3-v^2) - i\frac{(3-v^2)^2}{90}\xi\mu^2 t^5 + \frac{\mu^2 t^2}{60}(15 - 2v^2 + 3v^4)\right],$$

where

$$\xi = \frac{[(1-v^2)\kappa]^2}{16}, \quad \kappa = 2\sqrt{r}\mu = \frac{\omega}{m}\mu. \quad \text{(C.3)}$$

We will use the known integrals

$$\int_{-\infty}^{\infty}\cos\left(t + \xi\frac{t^3}{3}\right) = \sqrt{3}zK_{1/3}(z), \quad z = \frac{2}{3\sqrt{\xi}} = \frac{8}{3(1-v^2)\kappa},$$

$$\int_{-\infty}^{\infty} t\sin\left(t + \xi\frac{t^3}{3}\right) = \frac{3\sqrt{3}}{2}z^2 K_{2/3}(z). \quad \text{(C.4)}$$

Differentiating the first integral over ξ one gets

$$\int_{-\infty}^{\infty} t^3 \sin\left(t + \xi\frac{t^3}{3}\right)$$
$$= \frac{27\sqrt{3}}{8}z^3\frac{d}{dz}(zK_{1/3}(z)) = -\frac{27\sqrt{3}}{8}z^3\left(zK_{2/3}(z) - \frac{2}{3}K_{1/3}(z)\right),$$

$$\xi\int_{-\infty}^{\infty} t^6 \cos\left(t + \xi\frac{t^3}{3}\right) = -\frac{81\sqrt{3}}{16z^2}\left(z^3\frac{d}{dz}\right)^2(zK_{1/3}(z))$$

$$= \frac{81\sqrt{3}}{16}z^3\left[4zK_{2/3}(z) - \left(\frac{16}{9} + z^2\right)K_{1/3}(z)\right]. \quad \text{(C.5)}$$

Substituting the integrals Eqs.(C.3)-(C.4) in Eq.(C.2) we obtain the proba-

bilities of pair creation in standard quasiclassical approximation

$$W_2 = -\text{Im}\frac{\kappa_2}{\omega} = \frac{\alpha m^2}{3\sqrt{3}\pi\omega}\int_{-1}^{1}\frac{3+v^2}{1-v^2}K_{2/3}(z)dv,$$

$$W_3 = -\text{Im}\frac{\kappa_3}{\omega} = \frac{2\alpha m^2}{3\sqrt{3}\pi\omega}\int_{-1}^{1}\frac{3-v^2}{1-v^2}K_{2/3}(z)dv, \qquad (C.6)$$

and the corresponding corrections to this approximation

$$W_i^{(1)} = \frac{\alpha m^2\mu^2}{30\sqrt{3}\pi\omega\kappa}\int_{0}^{1}G_i(v,z)\frac{dv}{1-v^2}, \qquad (C.7)$$

where

$$G_2(v,z) = (36+4v^2-18z^2)K_{1/3}(z) + (3v^2-57)zK_{2/3}(z),$$

$$G_3(v,z) = -(34+2v^2+36z^2)K_{1/3}(z) + (78-6v^2)zK_{2/3}(z). \,(C.8)$$

In derivation of Eqs.(C.7)-(C.8) the integration by parts was used as well as the intermediate equalities in Eq.(C.5). The obtained representation of probabilities is convenient for calculation asymptotic at $\kappa \gg 1$ which are

$$W_i^{(1)} = \frac{\alpha m^2\mu^2}{30\sqrt{3}\pi\omega\kappa}w_i, \quad w_2 = 12A\kappa^{1/3}-90\pi,$$

$$w_3 = -11A\kappa^{1/3}+84\pi, \quad A = 3^{1/3}\frac{2}{5}\frac{\Gamma^3(1/3)}{\Gamma(2/3)} = 8.191... \qquad (C.9)$$

At $\kappa \ll 1$ the main contribution is given by the terms $\propto z^2$ in G_i Eq.(C.8). One has

$$W_2^{(1)} = \frac{\alpha m^2\mu^2}{\omega\kappa^2}\frac{2\sqrt{2}}{5\sqrt{3}}\exp\left(-\frac{8}{3\kappa}\right), \quad W_3^{(1)} = 2W_2^{(1)}. \qquad (C.10)$$

The relative magnitude of the corrections at $\kappa \gg 1$

$$\frac{W_2^{(1)}}{W_2} \simeq \frac{2\mu^2}{\kappa^{4/3}}\left(1-\frac{2.9}{\kappa^{1/3}}\right), \quad \frac{W_3^{(1)}}{W_3} \simeq \frac{11\mu^2}{9\kappa^{4/3}}\left(\frac{2.9}{\kappa^{1/3}}-1\right), \qquad (C.11)$$

At $\kappa \ll 1$ one has

$$\frac{W_i^{(1)}}{W_i} \simeq -\frac{32\mu^2}{15\kappa^3}, \qquad (C.12)$$

what agrees with Eq.(B.7).

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D Appendix

On the derivation of created particle spectrum in SQA

Substituting in Eq.(4.3) $x = (1+v)/2$ one has

$$\frac{dW^{SQA}}{dv} = \frac{\alpha m^2}{2\sqrt{3}\pi\omega}\left[2\frac{1+v^2}{1-v^2}K_{2/3}(z) + \int_{z}^{\infty}K_{1/3}(y)dy\right], \qquad (D.1)$$

where z is defined in Eq.(3.2). If one omits the integration over v in Eq.(3.7), the result of this manipulation disagrees with Eq.(D.1). So the described manipulation is erroneous. Only if one performs integration by parts of the second term in Eq.(D.1) and use the recursion relation

$$zK_{1/3}(z) = -z\frac{dK_{2/3}(z)}{dz} - \frac{2}{3}K_{2/3}(z), \qquad (D.2)$$

one obtains Eq.(3.7) for integral probability.

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V.N. Baier and V.M. Katkov

**Pair creation by a photon
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В.Н. Байер, В.М. Катков

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в сильном магнитном поле
при высокой энергии**